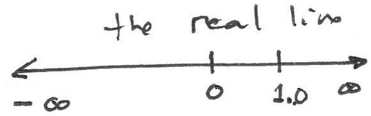
Before we proceed with $A\vec{n} = \lambda \vec{n}$, $\lambda = +1$.

a detour into $\mathbb{C}$:

- The real numbers $\mathbb{R}$ 

a set closed under addition & multip.

any 2 $\mathbb{R}$ under these operations is a real number again.

Field Properties : ① Closure under the operations $a, b \in \mathbb{R}$ then $a+b$, $a \cdot b \in \mathbb{R}$

② Both operations are associative.

$(a+b)+c = a+(b+c)$

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$

→ ③ Both operations are commutative :

$a+b = b+a$

$a \cdot b = b \cdot a$

④ $\exists$ an identity for addition $a+0 = 0+a \ \forall a \in \mathbb{R}$

⑤ $\exists$ an id. for mult. say 1, $a \cdot 1 = 1 \cdot a = a$

⑥ $\exists$ an inverse for addition, $\forall a \in \mathbb{R}$

$\exists \ -a \mid a + (-a) = 0$.

⑦ $\exists$ an inverse for multip, $\forall a \in \mathbb{R}$, $a \neq 0$

$\exists$ a number $\in \mathbb{R}, a^{-1} \mid a^{-1} \cdot a = a \cdot a^{-1} = 1$.

⑧ Multip. is distributive over addition :

$\forall a, b, c \in \mathbb{R} \rightarrow a \cdot (b+c) = a \cdot b + a \cdot c$.

These properties for ordinary arithmetic of real numbers were thought to hold for all algebraic structures but Hamilton's quaternions were a consistent algebraic system yet violated # 3 (commutativity) of multip.

Complex Numbers: Gauss (XIX Century)

- Motivation: The square of a real is always positive $a^2 \geq 0$

$x^2 + b^2 = 0 \rightarrow x^2 = -b^2 \rightarrow x = \sqrt{-b}$ cannot hold in $\mathbb{R}$.

more generally $x^2 - 2ax + a^2 + b^2 = 0$ $\forall$ $a, b \neq 0$ could not be resolved.

The numbers that satisfy these are Complex number.

$x = \pm ib$ $\quad$ i the imaginary number

$x = a \pm ib$ $\quad$ $\sqrt{-1} = i \Rightarrow i^2 = -1$.

$a + ib$, a real part
b imaginary part.

We identify the pair (a,b) with $a+ib$.

Equality for complex numbers: (a,b) an (c,d) are equal iff $\underline{a=c}$ & $\underline{b=d}$. Iff Their real parts are equal & their imaginary parts are equal.

Define the Sum & the product of 2 $\mathbb{C}$.

$$(a,b) + (c,d) = (a+c, b+d) \quad (1)$$

$$(a,b) \cdot (c,d) = (a \cdot c - bd, ad + bc) \quad (2)$$

We relate this ordered pair format to $a + ib$ notation.

$(a, 0) \longleftrightarrow a$
$(0, 1) \longleftrightarrow i$

Using (1) & (2) above:

$$i^2 = \underbrace{(0,1)}_{i} \cdot \underbrace{(0,1)}_{i} = (0.0 - 1.1, 0.1 + 1.0) = (-1, 0) = -1$$

$$i^2 = -1$$

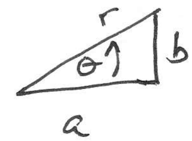
so that $(0,1) = i = \sqrt{-1}$

We may then write $(a,b) = (a,0) + (0,1)(b,0) = a + ib$.

Multiplication:

The magnitude of the product = the product of the magnitudes of the factors: and the angle of the product is the sum of the angles of the two factors.

$z = (a, b)$, $|z| = r$, $\arg(z) = \theta$.

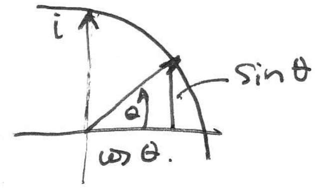


$a = r\cos\theta$

$b = r\sin\theta$

$z = (a, b) = a + ib$

$= r\cos\theta + i r\sin\theta = r(\cos\theta + i\sin\theta)$



by def. $e^{i\theta} = \cos\theta + i\sin\theta$

$z = re^{i\theta}$

r is magnitude (amplitude)

θ angle (or argument)

$z_1 = r_1(\cos\alpha + i\sin\alpha) = r_1 e^{i\alpha}$

$z_2 = r_2(\cos\beta + i\sin\beta) = r_2 e^{i\beta}$

then $\boxed{z_1 z_2 = r_1 r_2 e^{i(\alpha+\beta)}}$

$= r_1 r_2(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = r_1 r_2(\cos\alpha\cos\beta + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + i^2\sin\alpha\sin\beta)$

$= r_1 r_2[(\underbrace{\cos\alpha\cos\beta - \sin\alpha\sin\beta}) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)]$

$= r_1 r_2[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$

$\boxed{= r_1 r_2 \; e^{i(\alpha+\beta)}}$

Magnitudes are multiplied while angles are added.

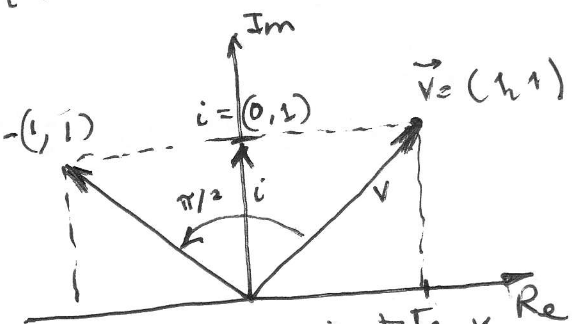
Let's now limit ourselves to complex numbers of magnitude 1.

then multip. by a $\mathbb{C}$. amounts to a rotation in the plane.

e.g. $\vec{v} = (1, 1)$, multip. by vector $i = (0, 1)$

$|i| = |(0,1)| = \sqrt{0^2 + 1^2} = 1$.

$\arg(i) = \frac{\pi}{2}$ rad.



Multiplying v by i rotates v

Subtraction

$(a+ib) + (c+id) = (a+c) + i(b+d)$

$(a+ib) - (c+id) = (a-c) + i(b-d)$

Multiplication

$(a+ib) \times (c+id) = ac + ibc + iad + \underline{\underline{i^2}}\, bd$

$= (ac - bd) + i(bc + ad)$

- A real number is called a scalar (as usual)

if set $b=0$ scalar a & $\mathbb{C}$ number $(c+id)$:

$a(c+id) = ac + iad$

Multip. by a scalar.

Quotient: rule $(c+id)(c-id) = c^2 + d^2$ is always a real number

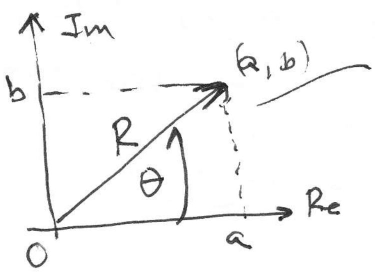
$$\frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} \cdot \frac{(c-id)}{(c-id)} = \frac{ac+bd + i(bc-ad)}{c^2+d^2} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

The reciprocal of $c+id$; $\frac{1}{c+id} = (c+id)^{-1} = \frac{c}{c^2+d^2} - i\frac{d}{c^2+d^2}$

$$\frac{1}{c+id} \cdot \frac{c-id}{c-id}$$

Polar Representation.

Identify the $\mathbb{C}$ (a,b), $a+ib$ with the poin (a,b) in the coordinatized plane.



Geometric Interpret.

Identify $z = (a,b)$ with the vector (a,b) from the origin O to (a,b)

$z = (a,b)$ define a magnitude

$|z| = \sqrt{a^2+b^2}$

and an angle called the argument. $\arg(z) = \theta$

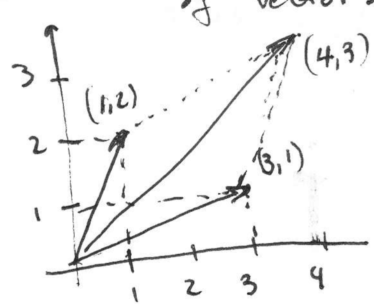
$\tan\theta = \frac{b}{a}$ $\begin{matrix} \to \sin\theta \\ \to \cos\theta \end{matrix}$

e.g. $z = (1,1)$

$|z| = \sqrt{2}$

$\theta = \arctan(1) = 45^\circ = \frac{\pi}{4}$ rad

Geometically Addition & Multip of $\mathbb{C}$ correspond to exactly that of vectors.

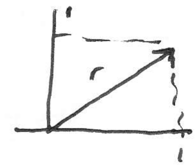


Parallelogram Rule

Vector Addition.

$z_1 = (1,1) \longrightarrow \quad |z_1| = \sqrt{1^2+1^2} = \sqrt{2}$

$z_2 = (0,1) \qquad |z_2| = 1$



$\arg(z_2) = \frac{\pi}{2}$

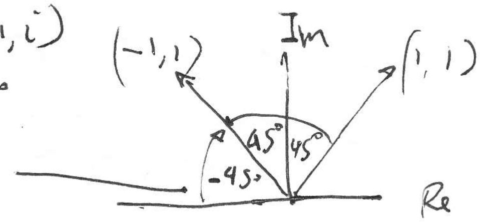
$\arg(z_1) = \frac{\pi}{4}$

$\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \cdot 1\,(0 + i)$

$(1+i)(i) = (i-1) = -1 + i) = (-1, i)$ $\quad -45°$

$(\sqrt{2})(1)\, e^{i\left(\frac{\pi}{2}+\frac{\pi}{4}\right)} = \sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)}$



Geometric interpret. of the product of $\mathbb{C}$ numbers: rotation of vector in the plane or a transformation of points in the plane.

$(1,1) \cdot \underbrace{(0,1)}_{i} = (-1,1)$ a rotation by $90°$.

The algebra of $\mathbb{C}$ numbers & the geometry of the Plane.

Hamilton $\rightarrow$ Is there an analogous relation in 3D?

can we use triplets of $\mathbb{R}$ numbers?

i, j, k. where $i^2 = j^2 = k^2 = ijk = -1$

- complex numbers of rank 4 – Quaternions.
- The set of all Quaternions along with the 2 operations of addition & multiplication form a RING.

  A non-commutative division ring.

  $\downarrow$ multip. inverse exists $\forall\ q \neq 0$.

- They satisfy all field properties except for commut. multip.