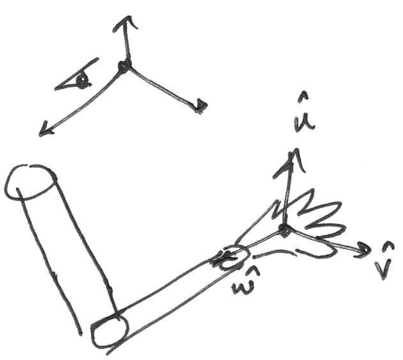
<u>Objective:</u> To describe the rotations of a rigid body (to which we anchor a 3Dim. coordinate axis) with one point fixed described by a rotation about some axis. We can track consequtive changes in orientation of the rigid body by tracking the positions of the 3 axes. This requires 3 degrees of freedom (DoF) according to <u>Euler's Theorem</u>.

An Example where we can use this is to track the orientation of the hand as the hand approaches an object to grasp it. We can represent the hand's orientation using a <u>rotation matrix</u> formed by a triad $\hat{u}, \hat{v}, \hat{w}$ of orthogonal unit vectors fixed on the hand. We can track in time the orientation of this triad and hence the orientation of the hand relative to the reference coordinate system (e.g. the observer's coord. system anchored somewhere, like the eyes)



In this case we represent the orientation-point by a matrix A

$$A = \begin{bmatrix} \hat{u}_x & \hat{v}_x & \hat{w}_x \\ \hat{u}_y & \hat{v}_y & \hat{w}_y \\ \hat{u}_z & \hat{v}_z & \hat{w}_z \end{bmatrix}$$

Scalar-matrix entry

unit vector. (col)

Properties of this matrix A (orthogonal):

1- It is a real, orthogonal matrix (each col. repres. a unit vector)

2- The eigenvalues of A are $\{1, e^{\pm i\theta}\} = \{1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

with i the imaginary unit, $i^2 = -1$.

3- $\det(A) = |A| = +1$, which is the product of its eigenvalues

$$\det(A) \begin{bmatrix} 1 & & 0 \\ & \cos\theta + i\sin\theta & \\ 0 & & \cos\theta - i\sin\theta \end{bmatrix} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i\cos\theta\sin\theta + i\,\cancel{\sin\theta}\cos\theta + \sin^2\theta = 1$$

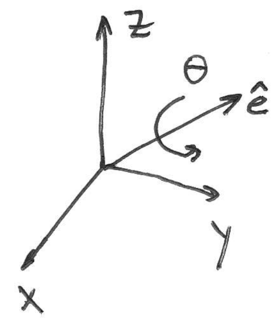
det(A) ↓ diagonalized

(by Pythagoras)

4- The Trace of A $Tr(A) = 1 + 2\cos\theta$, the sum of its eigenvalues

The angle θ is the angle of the Euler axis-and-angle represent.

The eigenvector of unit length corresponding to the eigenvalue of $+1$ is the Euler axis. This axis is the non-zero vector (the only one) which remains unchanged by left-multiplying (rotating) it with the rotation matrix.



$\theta \rightarrow$ angle of rotation (Euler angle), CCW positive.

$\hat{e} \rightarrow$ unit vector (Euler axis)

Summary of vector-col properties:

1- $|\hat{u}| = |\hat{v}| = 1$ (unit vectors)

2- $\hat{u} \cdot \hat{v} = 0$ (dot product) $\rightarrow$ orthogonal

3. $\hat{w} = \hat{u} \times \hat{v}$ mutually orthogonal form a frame, linearly indep., etc.

θ - scalar

$\hat{e}$ unit vector

$\hat{e} = [e_x\ e_y\ e_z]^T$

Notes of the representation:

- The normalized axis amounts to 2 DoF. The 3rd DoF. is the angle
- In some contexts one represents a rotation "vector"

$\vec{v} = \theta\hat{e}$ i.e. a non-normalized 3-dim vector, the direction repres. the axis and the magnitude is the θ. useful (economical) because only 3 scalars repres. the rotation BUT if $\theta = \emptyset$ the rot. is not uniquely defined. Combining successive rotations is troublesome then, as the "vectors" do not satisfy the law of vector addition.

Motivation to introduce quaternions as rotation operators.

Finite rotations are not vectors at all, better to employ the rotation matrix, turn it to quaternion notation, calculate the product, then turn it to Euler-angle-axis-representation

$$[A] \rightsquigarrow \hat{q} \rightsquigarrow \otimes \rightsquigarrow R(\theta, \hat{e}) \rightsquigarrow [A]$$

Let's obtain the Euler angle-vector-repres.

Given $A \in SO(3)$ determine the angle, vector repres.

Notation: $R \to$ repres. the matrix
$\phi \to$ repres. the angle
$\vec{n} \to$ repres the unit vector
$A \to$ will use it as a generic matrix (with some properties convenient to save time & effort).

Sources: "Rotations, Quaternions and Double groups" by Simon L. Altmann (Clarendon Press, Oxford 1986).

A parametrization of a rotation in the form $R(\phi, \vec{n})$

① ϕ must be in the accepted range $(-\pi, \pi)$

② $\vec{n}$ is unit length

③ $R(\phi, \vec{n}) \equiv R(-\phi, \vec{n})$

Known → • Every real orthogonal matrix leaves lengths & angles invariant. It must correspond to a rotation. (we assume this here).

• Its det is $+1$ (proper rotations) $SO(3)$.

Problem Given a special, real, orthogonal matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

determine $\vec{n}$ and ϕ.

Angle ϕ → Relates to $Tr(A)$ (the sum of the diag.)
The trace A invariant under similarity transformation
A must transform into a matrix like $SO(2)$

$\hat{R}(\alpha \vec{z})$ $\begin{bmatrix} \begin{bmatrix} C_\alpha & S_\alpha \\ -S_\alpha & C_\alpha \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$ → recall our examples.

Under the similarity the z-axis is chosen to coincide with the Euler axis of rotation (and it is a unit vector).

$$\hat{R}(\alpha \vec{z}) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After the similarity transformation A transform into some matrix like $\hat{R}(\alpha \vec{z})$ with α as ϕ.

Same Trace as Tr A:

$$\boxed{\cos\phi = \frac{1}{2}(\text{Tr } A - 1)}$$

Recall Similarity transform. & the properties of det.

$A = P^{-1}. B. P \rightarrow$ P is the change of basis matrix

B is the similar to A. matrix

From our example in class:

$$\underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A \text{ (Frame rotates)}} \text{ passive} \qquad \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A' \text{ (vector rotates)}} \text{ active}$$

$$|A'| = |B A B^{-1}| = |B| . |A| . |B^{-1}| = |B| . |A| . \frac{1}{|B|} = |A|$$ for some B

$|\,|B| \neq 0$ (invertible).

The same det. Take the trace, it is the same trace as well.

Find the axis of rotation $\vec{n}$.

$\vec{n}$ is the the eigenvector corresponding to the eigenvalue +1.

Need to use properties of the eigenvalues and eigenvectors of orthogonal matrices:

1- $A\vec{r} = a\vec{r}$

2- $\vec{r}^T A^T = a\vec{r}^T$

3- $\vec{r}^T A^T A \vec{r} = a^2 \vec{r}^T \vec{r}$

4- $\vec{r}^T \vec{r} = a^2 \vec{r}^T \vec{r}$

$\vec{r}$ vector

$\vec{r}^T$ vector transpose

A matrix

a scalar

cannot cancel $\vec{r}^T\vec{r}$ here because only if $\vec{r}$ is Real $\vec{r}^T\vec{r} \neq 0$. so in gral. cannot divide by it.

The real eigenvectors of an orthogonal matrix must correspond to eigenvalus $|\, a^2 \equiv$ unity.

$1^2 = 1$

$-1^2 = 1$

Since $A = A^*$, $\underline{A \text{ orthogonal}}$ (A^* is the conjugate of A)

$A^* \vec{r}^{\,*} = a^* \vec{r}^{\,*} \Rightarrow A \vec{r}^{\,*} = a^* \vec{r}^{\,*}$

take the transpose of A, then negate the imaginary part of the complex entries (but not the real parts) these are the complex conjugates.

e.g. $A = \begin{bmatrix} 1+i & 2 \\ 3-i & 4 \end{bmatrix}$, $A^* = \begin{bmatrix} 1-i & 3+i \\ 2 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 1+i & 3-i \\ 2 & 4 \end{bmatrix}$

from $A\vec{r} = a\vec{r}$

and $A\vec{r}^{\,*} = a^* \vec{r}^{\,*}$

then for real orthogonal matrices eigenvalues and eigenvectors come in conjugate pairs.

If $\vec{r}$ is real, $A\vec{r} = a^* \vec{r}$

On equating $\begin{cases} A\vec{r} = \underline{a}\,\vec{r} \\ A\vec{r} = \underline{a^*}\,\vec{r} \end{cases}$ it follows that the corresponding eigenvalue must also be real.

Thus applying $\boxed{\vec{r}^{\,T} \cdot \vec{r} = a^2\, \vec{r}^{\,T} \cdot \vec{r}}$

$\downarrow$

The real eigenvectors of a real orthogonal matrix must correspond to eigenvalues ± 1.

- These results apply to orthogonal matrices $O(3)$ (eventually we want to work in $SO(3)$, $\det = +1$, eigenvalue $+1$.) representing proper rotations.
- Because the eigenvalues of $O(3)$ matrices come in conjugate pairs, we have 2 possible cases.

(a) 1 real, 2 complex eigenvectors (conjugate)

(b) 3 real eigenvectors.

we use $A\vec{r}^{\,*} = a^* \vec{r}^{\,*}$ (the properties above)

$A\vec{r} = a^* \vec{r}$

(a1) $a_1 = 1$, $a_2 = w$, $a_3 = w^*$ $\Rightarrow$ $\det A = w w^*$

e.g. $\begin{bmatrix} \underbrace{1}_{a_1} & 0 & 0 \\ 0 & \underbrace{e^{i\theta}}_{a_2} & 0 \\ 0 & 0 & \underbrace{e^{-i\theta}}_{a_3} \end{bmatrix}$ $\det = a_1 \cdot a_2 \cdot a_3$ $\downarrow$ $\det = +1$.

Proper rotation

(a_2) $a_1 = -1$

$a_2 = w$

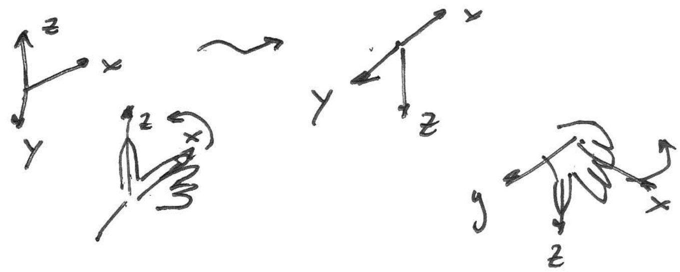
$a_3 = w^*$

$\underline{\det = -1}$ → improper rotation

(roto reflexion)

axis of rot. corresp. to eigenvalue -1.

the corrsp. eigenvector is reflected by this operation. e.g. changing a right-handed system into a left-handed

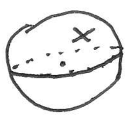


curl the fingers from x to y ccw

3 real eigenvectors.

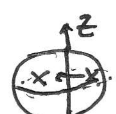
(b) We have 4 cases

(b_1) $a_1 = 1$
$a_2 = 1$
$a_3 = 1$
$\det A = 1$ (identity)



Same point left invariant

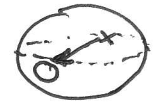
(b_2) $a_1 = 1$
$a_2 = 1$
$a_3 = -1$
$\det A = -1$
(reflexion)



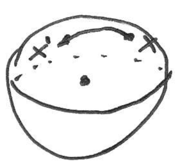
Point x passes to the opposit hemisp. (same latitude)

(b_3) $a_1 = 1$
$a_2 = -1$
$a_3 = -1$
$\det A = 1$
(binary rotation)

(b_4) $a_1 = -1$
$a_2 = -1$
$a_3 = -1$
$\det A = -1$
(inversion)



x Point on the upper hemisphere goes to the lower hemisp.



rotation on unit sphere (det + 1)

These are symmetry operations. Notice that we distinguish from rotations (+1) & binary rotations (+1) and the identity (+1) in that rotations satisfy $\underline{\det A = ww^*}$ they have only 1 real eigenvector

1 eigenvalue real & 2 conjugate complex.

$$\begin{bmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{bmatrix}$$

Determine $\vec{n} = (n_x \quad n_y \quad n_z)$ the Euler axis corresp. to eigen value $+1$. (the other 2 eigenvalues being complex conjugates).

- Find the solution of the characteristic equation for A corresp. to root $+1$.
- A shortcut by Altmann.

A real-symmetric matrix (even an orthogonal one) cannot be a general rot. matrix. Yet every matrix can be written as a sum of a symmetric and a skew-symmetric

$$A = \frac{1}{2}\Big[\underbrace{\{A + A^T\}}_{\text{Symmetric}} + \underbrace{\{A - A^T\}}_{\substack{\text{skew sym.}\\ S}}\Big]$$

e.g. $\frac{1}{2}\left\{\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} + \begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}\right.$

$\frac{1}{2}\begin{bmatrix}2 & 6 & 10\\ 6 & 10 & 14\\ 10 & 14 & 18\end{bmatrix} = \begin{bmatrix}1 & 3 & 5\\ 3 & 5 & 7\\ 5 & 7 & 9\end{bmatrix}$

A symmetric matrix

S is the skew-symm. component.

$S = A - A^T \rightarrow$ very important in establishing the properties of rotations.

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} - \begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix} = \begin{bmatrix}0 & -2 & -4\\ 2 & 0 & -2\\ 4 & 2 & 0\end{bmatrix}$$

a Skew-Symmetric one.

$$\begin{bmatrix}0 & A_{12}-A_{21} & A_{13}-A_{31}\\ A_{21}-A_{12} & 0 & A_{23}-A_{32}\\ A_{31}-A_{13} & A_{32}-A_{23} & 0\end{bmatrix}$$

$$=_{def} \begin{bmatrix}0 & c & b\\ -c & 0 & a\\ -b & -a & 0\end{bmatrix}$$

We connect S with the eigenvector equation of A (since its eigenvalue is unity)

$$A\vec{n} = (1)\vec{n}$$

From the orthogonality condition, $AA^T = Id$.

$$\vec{n} = A^T . \vec{n}$$

$$A\vec{n} = A^T . \vec{n}$$

$$(A - A^T)\vec{n} = \vec{0} \quad \Rightarrow \quad S\vec{n} = \vec{0}$$

We use def above:

$$S\vec{n} = \begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \vec{0}$$

$$\begin{aligned} c n_y + b n_z &= 0 \\ -c n_x + a n_z &= 0 \\ -b n_x - a n_y &= 0 \end{aligned} \qquad \begin{aligned} n_x &= a n_z / c \\ n_y &= -b n_z / c \end{aligned}$$

n_x, n_y in terms of n_z

Use the normalization condition

$$\left(n_x^2 + n_y^2 + n_z^2 \right)^{1/2} = \left[\left(\frac{a n_z}{c}\right)^2 + \left(\frac{-b n_z}{c}\right)^2 + n_z^2 \right]^{1/2} = 1$$

$$\left[\frac{a^2}{c^2} n_z^2 + \frac{b^2}{c^2} n_z^2 + n_z^2 \right]^{1/2} = 1$$

$$c \left[a^2 + b^2 + c^2 \right]^{1/2} n_z = 1 \rightarrow \begin{cases} n_z = c\,(a^2+b^2+c^2)^{-1/2} \\ n_x = \frac{a}{\cancel{c}} \cdot \cancel{c}\,(a^2+b^2+c^2)^{-1/2} \\ n_y = -b\,(a^2+b^2+c^2)^{-1/2} \end{cases}$$

Notice that to obtain $(a^2+b^2+c^2)^{-1/2}$ we use the $Tr(SS^T)$. Multiply these matrices & you will obtain

$$(a^2+b^2+c^2) = \frac{1}{2} Tr(SS^T)$$

Also $SS^T = (A - A^T)(A^T - A) = 2\,Id - A^2 - (A^T)^2$

A and A^2 on diagonalization will have the form

	roots
1	+1
2	complex
3	conjugates

$$A' = \begin{bmatrix} 1 & & \\ & e^{i\phi} & \\ & & e^{-i\phi} \end{bmatrix} \qquad (A')^2 = \begin{bmatrix} 1 & & \\ & e^{i2\phi} & \\ & & e^{-i2\phi} \end{bmatrix}$$

same deal.

recall that arguments add up when you square. complex

$Tr\, A^2 = 1 + 2\cos 2\phi$ use trig ident.

$$\cos 2t = 2\cos^2 t - 1$$

and $\sin^2\theta = 1 - \cos^2\theta$

Detour

Some Properties of Tr

1. $Tr(A+B) = Tr(A) + Tr(B)$ } Linearity
2. $Tr(cA) = c\,Tr(A)$
3. Let $A_{m\times n}$, $B_{n\times m}$
   $Tr(AB) = Tr(BA)$
4. Similarity Invariant $\rightarrow$ $Tr(A) = Tr(P^{-1}AP) = Tr(P^{-1}(AP)) = Tr((AP)P^{-1}) = Tr(AI) = Tr(A)$
5. $Tr(A) = Tr(A^T)$

$\mathrm{Tr}(2\,Id) - \mathrm{Tr}(A^2) - \mathrm{Tr}(A^T)^2$

$4 - (1 + 2\cos\phi) - (1 + 2\cos 2\phi)$

$4 - 1 - 2\cos\phi - 1 - 2\cos 2\phi$

$2 - 4 \underbrace{\cos 2\phi}$

use $\left\{ \cos 2t = 2\cos t^2 - 1 \right.$

$2 - 4\underbrace{(2\cos\phi^2 - 1)}$

use: $\sin^2 = \cos^2 - 1$

$8\sin^2\phi$

recall

$(1 + e^{i2\phi} + e^{-i2\phi})$

$1 + (\cos 2\phi + i\sin 2\phi) +$

$(\cos 2\phi - i\sin 2\phi)$

$(1 + 2\cos 2\phi) \to (A')^2$

$(1 + \cos\phi) \longrightarrow (A')$

$\mathrm{Tr}(SS^T) = 8\sin^2\phi$ giving $a^2 + b^2 + c^2 = 4\sin^2\phi$.

$n_x = \pm a(2\sin\phi)^{-1}$, $n_y = \mp b(2\sin\phi)^{-1}$, $n_z = \pm c(2\sin\phi)^{-1}$

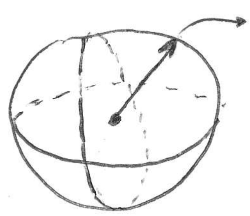
$n = (001)$ c is $A_{12} - A_{21}$ from the previous matrix.

On identifying explicitly a, b, c. with the entries of the matrix, we get.

$$n_x = \frac{A_{32} - A_{23}}{2\sin\phi} \qquad n_y = \frac{A_{13} - A_{31}}{2\sin\phi} \qquad n_z = \frac{A_{21} - A_{12}}{2\sin\phi}.$$

and $R(-\phi, -\vec{n})$ can be also obtained.

In Summary: We can reduce 9 numbers of a Rotation matrix in SO(3) as 3 Dof: 2Dof provided by the unit vector



Unit Sphere

which gives us azimuth & elevation on the unit sphere,

and 1 Dof provided by the angle of rotation