Quaternions: Hyper complex numbers of rank 4.

(real numbers are hyper complex numbers of rank 1, complex numbers on the complex plane are of rank 2)

The rank refers to the number of elements needed to define them.

1- Real numbers satisfy the field properties under ordinary addition and multiplication.

2- Complex numbers also satisfy the field properties.

3- Any complex numbers of rank > 2 do not satisfy the field properties.

4- $\boxed{i^2 = j^2 = k^2 = ijk = -1}$ hyper complex numbers of rank 4. (quaternions). They are very well suited to represent rotations.

The set of quaternions under the operations of addition and multiplication satisfy all of the axioms of a field except the commutative law of multiplication.

Definition:

notation q, p, r will represent quaternions.

$\vec{i}\ \vec{j}\ \vec{k}$ ordinary vectors in $\mathbb{R}^3$

$\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$

The standard basis of $\mathbb{R}^3$ forming an orthonormal basis

$q = (q_0, q_1, q_2, q_3)$ a 4-tuple of real numbers $\in \mathbb{R}^4$, $q_0, q_1, q_2, q_3 \in \mathbb{R}$

$\vec{q} = iq_1 + jq_2 + kq_3 \rightarrow$ vector part $\in \mathbb{R}^3$, q_1, q_2, q_3 the components of the quaternion.

$q_0 \in \mathbb{R} \rightarrow$ Scalar part $\in \mathbb{R}$

Quaternion: $q = q_0 + \vec{q} = q_0 + iq_1 + jq_2 + kq_3$

Quaternion Operations :

Equality : 2 quaternions are equal iff they have exactly the same components :

$p = p_0 + ip_1 + jp_2 + kp_3$

$q = q_0 + iq_1 + jq_2 + kq_3$

$$p = q \text{ iff } \begin{cases} p_0 = q_0 \\ p_1 = q_1 \\ p_2 = q_2 \\ p_3 = q_3 \end{cases}$$

Addition : Add component wise

$$p + q = (p_0 + q_0) + i(p_1 + q_1) + j(p_2 + q_2) + k(p_3 + q_3)$$

$p+q$ is a quaternion and satisfies the field properties for addition, so the set of quaternions is closed under addition.

$\emptyset$-quaternion $(0, 0, 0, 0)$.

q has a negative $-q$ (additive inverse)

$(-q_0, -q_1, -q_2, -q_3)$.

Addition is both associative and commutative (because addition of real numbers has these properties & they transfer).

Multiplication : If c is a scalar
q is a quaternion

$$c \cdot q = cq_0 + icq_1 + jcq_2 + kcq_3$$

Simply multiply each component of the quaternion by the scalar

→ The result is again a quaternion.
The set of quaternions is closed under multip. by a scalar

e.g. if $q = 3 + 2i + j + 4k$

$3q = 9 + 6i + 3j + 12k$.

Multiplication : 2 quaternions

Special (quaternion) products

$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = k = -ji$$
$$jk = i = -kj$$
$$ki = j = -ik$$

non-commutative

∴ the product of $p\cdot q \neq q\cdot p$ in general.

$$p = p_0 + ip_1 + jp_2 + kp_3$$
$$q = q_0 + iq_1 + jq_2 + kq_3$$

$$p\cdot q = (p_0 + ip_1 + jp_2 + kp_3)\cdot(q_0 + iq_1 + jq_2 + kq_3)$$

with some algebra:

$$\begin{aligned} p\cdot q = {} & p_0 q_0 - (p_1 q_1 + p_2 q_2 + p_3 q_3) \\ & + p_0(iq_1 + jq_2 + kq_3) + q_0(ip_1 + jp_2 + kq_3) \\ & + i(p_2 q_3 - p_3 q_2) + j(p_3 q_1 - p_1 q_3) + k(p_1 q_2 - p_2 q_1) \end{aligned}$$

Now recall the dot product & the cross product:

$$a = (a_1, a_2, a_3), \quad b = (b_1, b_2, b_3)$$

$$a\cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2 b_3 - a_3 b_2) + j(a_3 b_1 - a_1 b_3) + k(a_1 b_2 - a_2 b_1)$$

$$p\cdot q = p_0 q_0 - \vec{p}\cdot\vec{q} + p_0 q + q_0 p + p \times q$$

Working Definition of the product of 2 quaternions:

$$(\mathrm{I}) \quad \boxed{pq = p_0 q_0 - \vec{p}\cdot\vec{q} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}}$$

example: $p = 3 + i - 2j + k \;\rightarrow\; p = 3 + \vec{p}\,, \quad \vec{p} = (1,-2,1)$

$q = 2 - i + 2j + 3k \qquad q = 2 + \vec{q}\,, \quad \vec{q} = (-1,2,3)$

Calculate $\vec{p}\cdot\vec{q} = (1)(-1) + (-2)(2) + (1)(3) = -2$

$$\vec{p}\times\vec{q} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ -1 & 2 & 3 \end{vmatrix} = (-6-2)i - (3-(-1))j + (2-2)k = 8i - 4j$$

Using equation (I) above.

$$pq = 6 - (-2) + 3(-i + 2j + 3k) + 2(i - 2j + k) + (-8i - 4j)$$

$$= 8 - 9i - 2j + 11k.$$

The product defined in (I) may be written using the algebra of matrices:

$$pq = r = r_0 + \vec{r} = r_0 + ir_1 + jr_2 + kr_3$$

$$r_0 = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3$$

$$r_1 = p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2$$

$$r_2 = p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1$$

$$r_3 = p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0$$

$$\rightarrow \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\left.\begin{array}{l} p = p_0 + i p_1 + j p_2 + k p_3 \\ q = q_0 + i q_1 + j q_2 + k q_3 \end{array}\right\} r_0 \to + \; - \; - \; -$$

$$\left.\begin{array}{l} p = p_0 + i p_1 + j p_2 + k p_3 \\ q = q_0 + i q_1 + j q_2 + k q_3 \end{array}\right\} r_1 \to + \; + \; + \; -$$

$$\left.\begin{array}{l} p = p_0 + i p_1 + j p_2 + k p_3 \\ q = q_0 + i q_1 + j q_2 + k q_3 \end{array}\right\} r_2 \to + \; - \; + \; +$$

$$\left.\begin{array}{l} p = p_0 + i p_1 + j p_2 + k p_3 \\ q = q_0 + i q_1 + j q_2 + k q_3 \end{array}\right\} r_3 \to + \; + \; - \; +$$

Numerical Example:

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ -2 & 1 & 3 & -1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ -2 \\ 11 \end{bmatrix}$$

$\therefore pq = r = 8 - 9i - 2j + 11k$. (as before).

- quaternions satisfy the field properties for addition
- quaternions form a non-commutative division ring (they have an inverse for multiplication & satisfy all field properties for mult. except commutativity)

(6)

The product of 2 quaternions is another quaternion.

Scalar part $p_0 q_0 - \vec{p} \cdot \vec{q}$

vector part $p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q}$.

- Thus the set of quaternions is closed under multiplication as well as under addition. The product is also associative.
- Since $\vec{p} \times \vec{q} \neq \vec{q} \times \vec{p}$ in general, the product does not commute
- The identity for quaternion multiplication is:

$$q = 1 + \vec{0}$$

- Multiplication of quaternions is also distributive over addition
- Every non-zero quaternion does have a multiplicative inverse (see next).

The Complex Conjugate.

Let $q = q_0 + \vec{q} = q_0 + i q_1 + j q_2 + k q_3$ be a quaternion

Denote $q^* = q_0 - \vec{q} = q_0 - i q_1 - j q_2 - k q_3$ is its c.conjugate

$$(pq)^* = q^* p^*$$ → The c.conjugate of the product of quaternions equals the product of the individual complex conjugates in reverse order.

$$q + q^* = (q_0 + \vec{q}) + (q_0 - \vec{q}) = 2q_0$$, a scalar

The Norm. $N(q)$ or $|q|$, the length of q is defined by

$$N(q) = \sqrt{q^* q} \longrightarrow q^* q = q q^* = |q|^2$$

$N(q) = \sqrt{q^* q} \rightarrow$ Use the def. of the quaternion product. & the fact that for any vector q we have $q \times q = 0$.

$$N^2(q) = (q_0 - \vec{q}) . (q + \vec{q}) = q_0 q_0 - (-\vec{q}).(\vec{q}) + \cancel{q_0 \vec{q}} + \cancel{(-\vec{q})} q_0 + \underbrace{(-\vec{q}) \times \vec{q}}_{0}$$

$$= q_0^2 + \vec{q} \cdot \vec{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 = |q|^2 \longrightarrow q^* q = q \cdot q^* = |q|^2$$

Example $q = 2 - i + 2j + 3k$

$$N^2(q) = 2^2 + (1)^2 + 2^2 + 3^2 = 18 \Rightarrow N(q) = \sqrt{18}$$

Notice that this definition is the same as that for the length of a vector in $\mathbb{R}^4$, the same meaning as any Euclidean Norm.

- we work with unit quaternions, of length 1.

$$N^2(p \cdot q) = (pq) \cdot (pq)^* = p \underbrace{q \cdot q^*} p^* = p N^2(q) p^* = p p^* N^2(q)$$

$$= N^2(p) . N^2(q)$$

- The product of two unitary quaternions is again a unit. quat.

Inverse of a Quaternion

If we define q^{-1} the inverse, $q^{-1} q = q q^{-1} = 1.$ | pre/post q^*

If we use pre- and post- multip. by the complex conjugate q^*

$$q^{-1} \cdot q q^* = q^* q q^{-1} = q^*, \quad \text{Since } q q^* = N^2(q) \text{ we get}$$

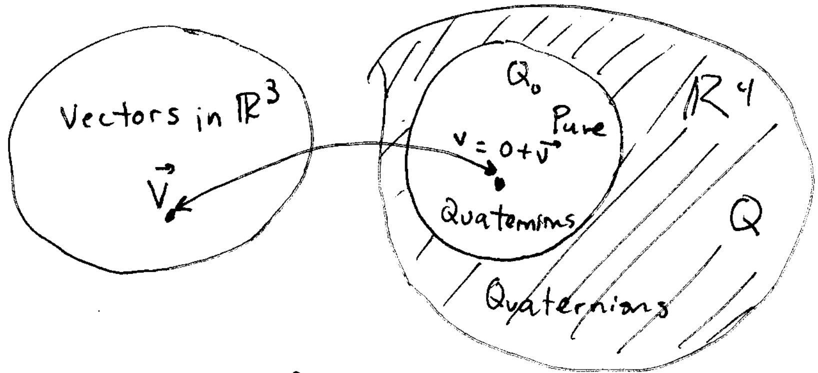
$$q^{-1} = \frac{q^*}{N^2(q)} = \frac{q^*}{|q|^2}$$

If q is unit-length then $q^{-1} = q^*$, analogous to $A^{-1} = A^t$ in rotation matrices.

Geometric Interpretation: Recall that if we have a vector $\vec{v} \in \mathbb{R}_3$ and a rot. matrix $A \in SO(3)$, $\vec{w} = A . \vec{v}$ is the rotated vector.

Identify a vector $\vec{v} \in \mathbb{R}^3$ with its pure quaternion $v = 0 + \vec{v}$

Question: How can a quaternion, which lives in $\mathbb{R}^4$ operate on a vector which lives in $\mathbb{R}^3$.



Correspondance

Vectors $\longleftrightarrow$ Quaternions

Isomorphism wrt. addition.

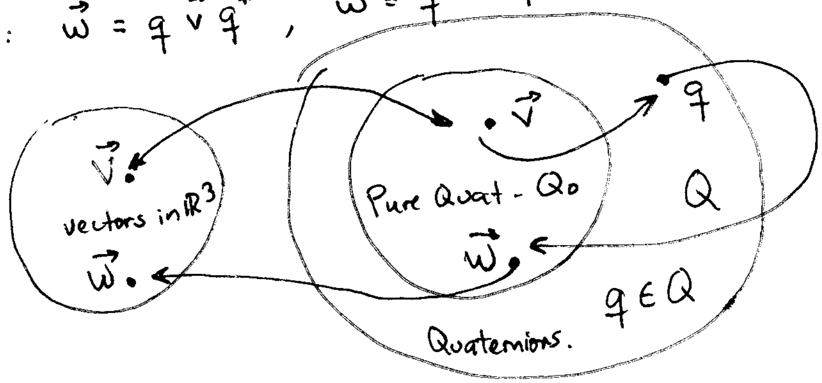
wanted $\underset{\substack{\uparrow \\ \text{image} \\ \text{(rotated vector)}}}{\vec{w}} = q \cdot \vec{v}$

recall quat. product. $pq = p_0 q_0 - \vec{p} \cdot \vec{q} + p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q}$

$$q\vec{v} = (q_0 + \vec{q})(0 + \vec{v})$$
$$= q_0 \cdot 0 - \vec{q} \cdot \vec{v} + 0\vec{q} + q_0 \vec{v} + \vec{q} \times \vec{v}$$
$$= \underbrace{-\vec{q} \cdot \vec{v}}_{\neq 0} + q_0 \vec{v} + \vec{q} \times \vec{v} \quad \notin \mathbb{R}^3, \notin Q_0$$

(and $\vec{v} \cdot q$ will not work either)

Triple product: $\vec{w} = q \vec{v} q^*$, $\vec{w} = q^* \vec{v} q$ Takes a vector into a vector.



Geometric Interpret.

Is there a way to associate an angle with a quaternion, analogous to the way that we associated an angle with the $R \in SO(3)$

$$q = q_0 + \vec{q} \qquad \text{(unit quat).} \qquad q_0^2 + |\vec{q}|^2 = 1.$$

since $\forall \theta$ we know $\sin^2\theta + \cos^2\theta = 1$

unit quat $\underbrace{q_0^2}_{1} + \underbrace{|\vec{q}|^2}_{\sin^2\theta} = \ldots$

We can identify $\cos^2\theta = q_0^2$ $\quad -\pi < \theta \leq \pi$

$\sin^2\theta = |\vec{q}|^2$

9)

Associate the angle θ with the quaternion q.

recall $\sin^2\theta = |\vec{q}|^2$
$\cos^2\theta = q_0^2$

$$q = q_0 + \vec{q} \rightarrow \underline{\text{define}} \text{ unit vector } \vec{u} = \frac{\vec{q}}{|\vec{q}|} = \frac{\vec{q}}{\sin\theta}$$

$\hookrightarrow \vec{q} = \vec{u} \cdot \sin\theta$

We write the unit quaternion q in terms of the angle θ and the unit vector $\vec{u}$ as

$$q = q_0 + \vec{q} = \cos\theta + \vec{u}\,\underline{\sin\theta}$$

recall $\sin(-\theta) = -\sin(\theta)$

using $(-\theta)$ we get the conjugate of q.

i.e. $\cos(-\theta) + \vec{u}\sin(-\theta) = \cos\theta - \vec{u}\sin\theta = q^*$

Suppose p, q are quaternions & let's express them using α, β

$$\left.\begin{aligned} p &= \cos\alpha + \vec{u}\sin\alpha \\ q &= \cos\beta + \vec{u}\sin\beta \end{aligned}\right\}$$

recall quat. prod. $pq = p_0q_0 - \vec{p}\cdot\vec{q} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}$.

$$r = pq = (\underline{\cos\alpha} + \vec{u}\sin\alpha)(\underline{\cos\beta} + \vec{u}\sin\beta) = \cos\alpha\cos\beta - (\vec{u}\sin\alpha)\cdot(\vec{u}\sin\beta) + \cos\alpha(\vec{u}\sin\beta) + \cos\beta(\vec{u}\sin\alpha) + u\sin\alpha \times u\sin\beta.$$

$$= \cos\alpha\cdot\cos\beta - \sin\alpha\sin\beta + \vec{u}\cdot(\sin\alpha\cdot\cos\beta + \cos\alpha\sin\beta)$$

$$= \cos(\alpha+\beta) + \vec{u}\sin(\alpha+\beta)$$

$$= \cos\gamma + \vec{u}\sin\gamma$$

recall trig id.
$\cos(\alpha+\beta) = cs - sc$
$\sin(\alpha+\beta) = sc + cs$

If we multiply the quat. p and q <u>with same vector $\vec{u}$</u> the prod. is a <u>quat.</u> $r = \cos\gamma + \vec{u}\sin\gamma$ also with vector $\vec{u}$ and angle $\gamma = \alpha + \beta$.

This is precisely what we should expect if quaternions represent rotations.

Special Quaternion Operator: $\vec{w} = q\,\vec{v}\,q^*$, $\vec{w} = q^*\,\vec{v}\,q$

- The norm of the product equals the product of the norm.

$$\vec{w} = q\,\vec{v}\,q^*$$

$$N(\vec{w}) = N(q\,\vec{v}\,q^*) = \underbrace{N(q)}\,.\,N(\vec{v})\,.\,N(q^*)$$

$$= 1\ .\ \underline{N(\vec{v})}\ .\ 1$$

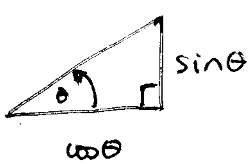
$$= N(\vec{v}) \qquad \text{(same for } \vec{w} = q^*.\,\vec{v}\,q\text{)}$$

- Vector-length invariant to the rotation as in $R \in SO(3)$.

- If we replace θ by $-\theta$ in $q = \cos\theta + \vec{u}\sin\theta$ we obtain the conjugate $q^* = \cos\theta - \vec{u}\sin\theta$

∴ By the appropriate choice of angle θ these operators may repres. the same geometric transformation.

<u>Question</u>: What geometric effect these operators have when applied to an arbitrary vector in $\mathbb{R}^3$?



$q = \cos\theta + k\sin\theta$

epsilon-incremental quaternion

e.g. suppose $u = 0i + 0j + 1k = \vec{k}$
$\{i, j, k\}$ standard basis in $\mathbb{R}^3$
say θ is very small. $\longrightarrow$ $\cos\theta \simeq 1$, $\sin\theta \simeq \theta$

then $q \simeq 1 + \underline{k\theta}$ $\left\{ 1 \text{ if } \theta = \emptyset \right.$

$$w = q\,i\,q^* = (1 + k\theta)(\emptyset + i)(1 - k\theta)$$ ← use the incremental quaternion in the operator $q\,\vec{v}\,q^*$ to determine its action on the basis $i \in \mathbb{R}^3$ (for ex.)

$$= (1 + k\theta)(\cancel{\theta}i + j\theta)$$

$$= \emptyset + i + 2\theta j$$

$\downarrow$

Pure quaternion.

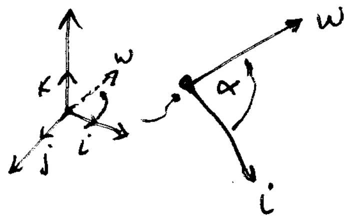
<u>recall that</u> $\left\{ \begin{array}{l} ij = k = -ji \\ jk = i = -kj \\ ki = j = -ik \leftarrow \\ i^2 = j^2 = k^2 = ijk = -1 \end{array} \right.$

Geom. Interpret. input vector $\vec{i}$ has been "tweaked" by the quat. operator $q\,i\,q^*$ to produce the output vector $\underline{w = \vec{i} + 2\theta j}$

θ^2 terms ignored as they're very small.

$$N(w) = \underbrace{N(q)}_{1}\ \underbrace{N(i)}_{1}\ \underbrace{N(q^*)}_{1} = 1.$$

Let α be the $\measuredangle$ bet. the input and output vectors in $\mathbb{R}^3$



$\tan\alpha = 2\theta$ $\left\{ \frac{\sin\theta}{\cos\theta} \right.$

$\tan\alpha \simeq \alpha$

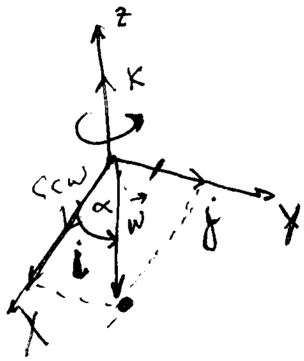
- for small α -

write $\alpha = 2\theta$

$\vec{w} = 0 + i + 2\theta j$

$\downarrow$

$\in$ Second quadrant of XY-plane in $\mathbb{R}^3$



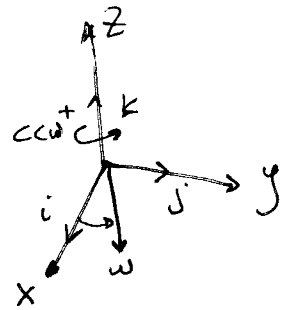
• $\vec{i}$ was rotated CCW. about $\vec{k}$ by an angle of 2θ

or

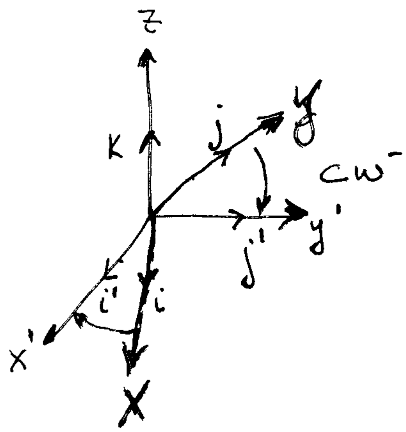
• the frame was rotated CW by -2θ.

Point-rotation

Coordinate Frame Fixed Perspective $\rightarrow$



$q \vec{v} q^*$



Frame-rotation

$\vec{v}$ fixed, frame rotates. $\underbrace{q^* \vec{v} q}$

Choose θ appropriately $-\pi < \theta \le \pi$.