ROTATIONS IN THE PLANE

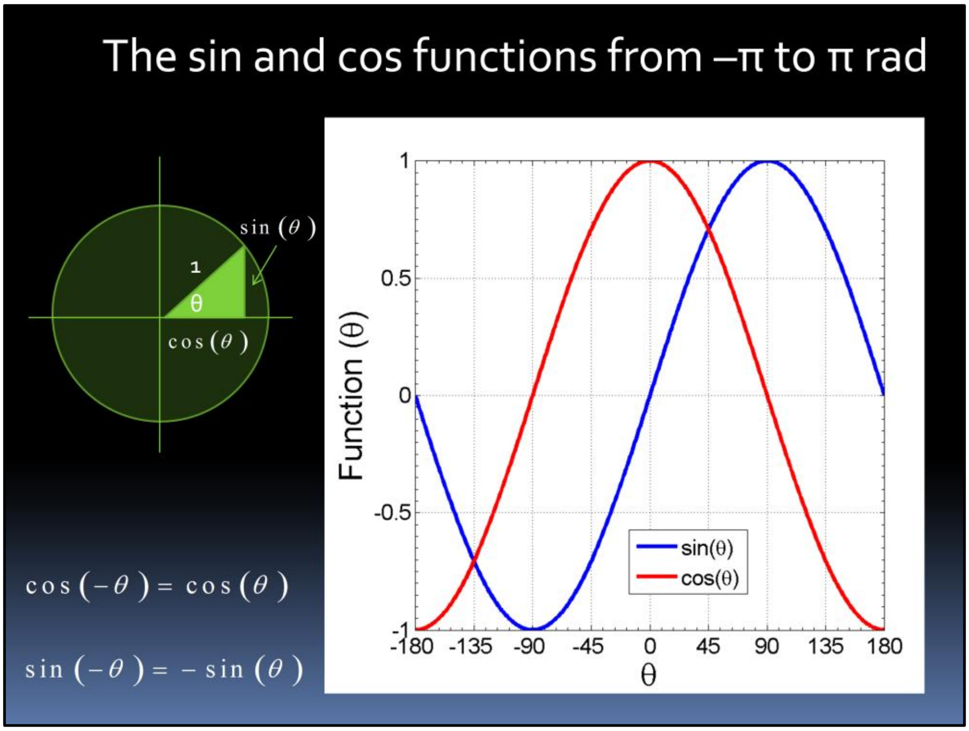
LECTURE NOTES WEEK 5 (10-06-11)

Elizabeth B Torres – Rutgers University Psychology
Integrative Methods in Perceptual Science

Textbooks

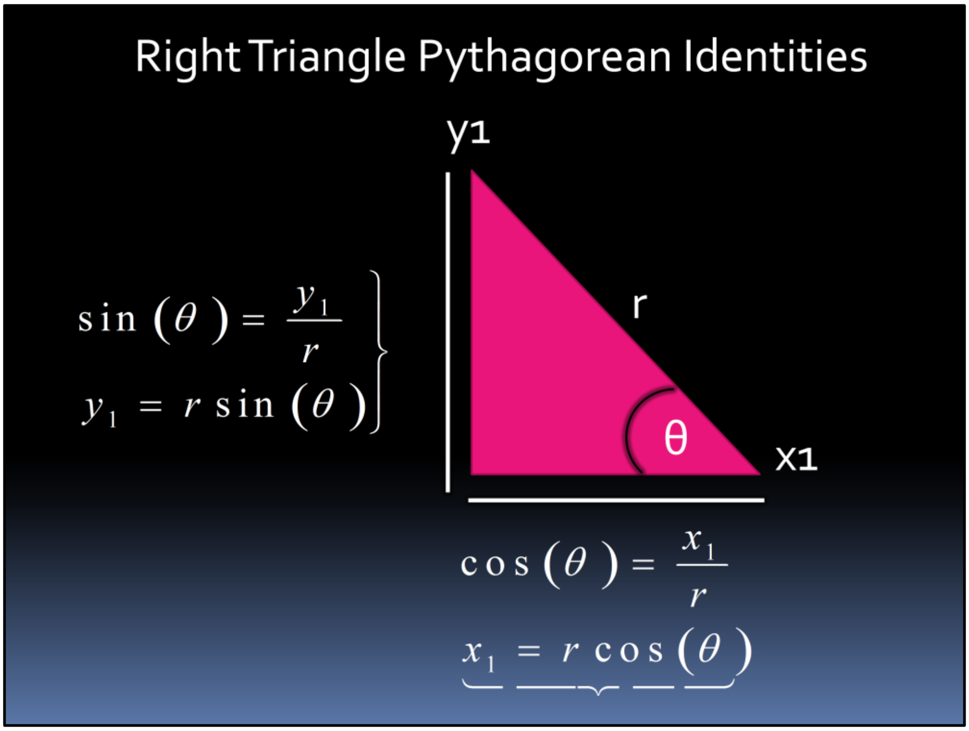
Quaternions and Rotation Sequences by J.B. Kuipers

Rotations, Quaternions and Double Groups by S. Altman



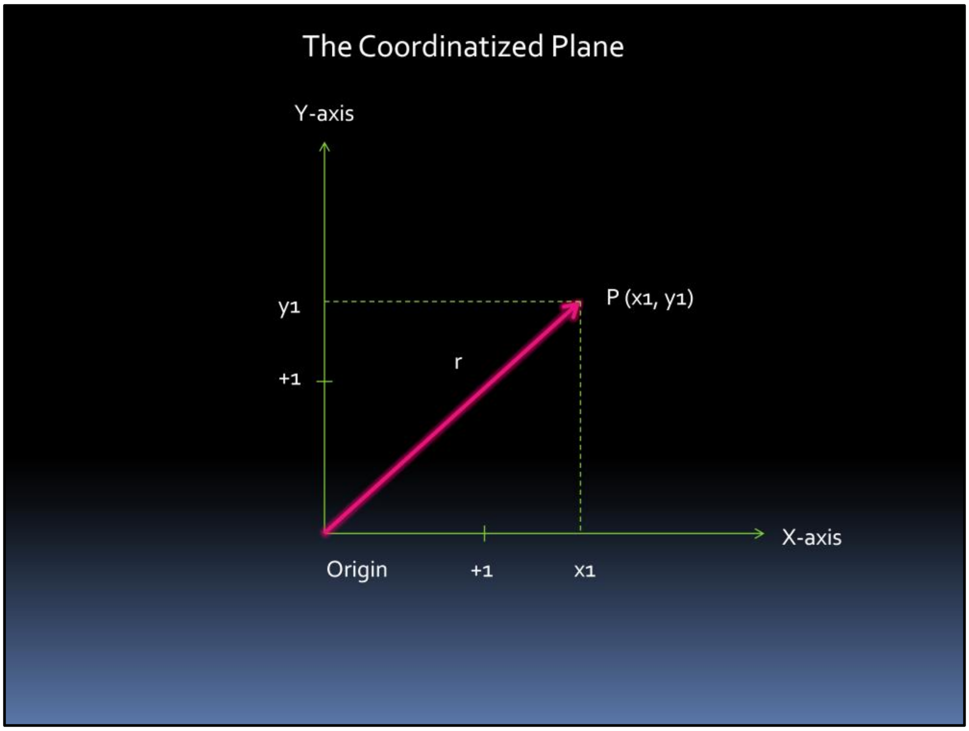
Let theta be the angle measured CCW (our positive convention) from the x-axis along the arc of the unit circle, the sin of theta is the vertical coordinate of the arc endpoint divided by the hypotenuse (which is 1), whereas the cos of theta is the horizontal side (base) of the triangle divided by the hypotenuse (which is 1)

Follow the values of theta and the function (sin or cosine) and see that the identities on the left hold. The graph is in degrees (just in case that you prefer degrees). To convert to radians multiply the degrees by π rad/180 deg, the interval here goes from –π to π



The sides of the right triangle are x1 (side), y1 (opposite) and r (hypothenuse)
The Pythagorean identities give us the cosine and the sine of the angle theta:
cos(theta) = side / hypothenuse whereas sin(theta) = opposite / hypothenuse

Now express x1 (side) and y1 (opposite) in terms of the respective cos and sine of the angle theta and the hypothenuse and you have the coordinate functions for the next page. We'll use these expressions throughout the class today.



Consider an object (or set of points on a rigid body) in a plane. The relative location of each point is defined wrt a coordinate frame fixed in the plane.

In the plane an arbitrary by fixed point (O for origin) will be specified to refer points to it.

Two straight lines intersecting perpendicularly will define our reference frame in the plane.

The x-axis is horizontal and the y-axis is vertical.

The origin is denoted (0,0). At a unit distance we mark a step along the x- and another along the y-axis.
These have coordinates (+1,0) and (0,+1) respectively. Now we can identify every point on the plane by an ordered tuple relative to this fixed origin. This provides a coordinatized plane denoted R^2

Rotations in the Plane

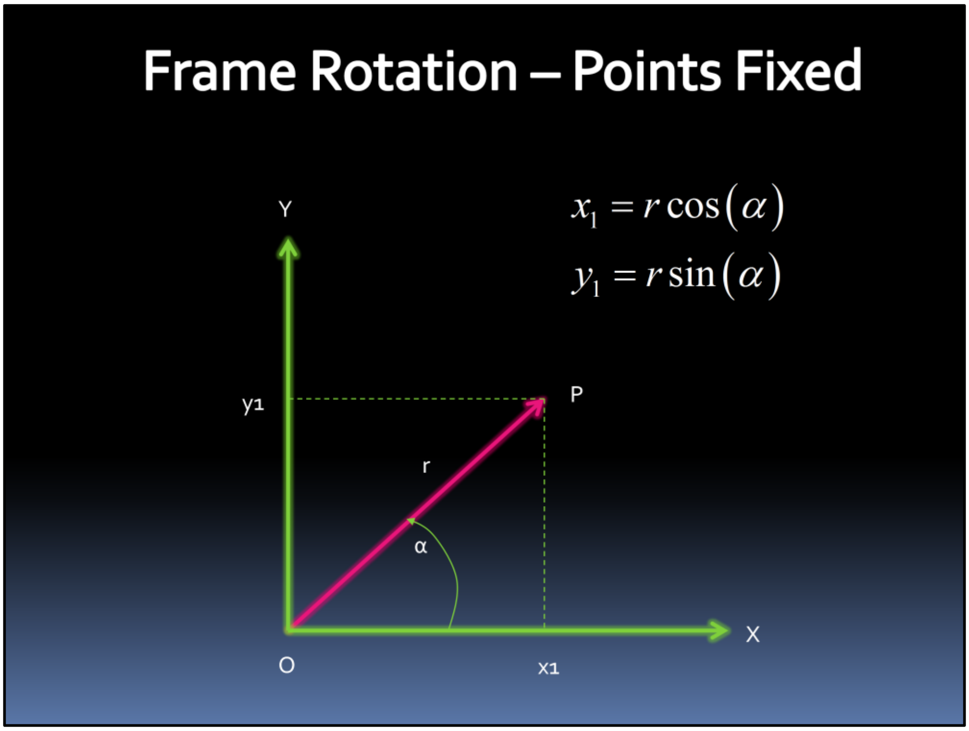
Two Perspectives:

Frame Rotations – Point Fixed

Point Rotation – Frame Fixed

We will first determine what effects the rotation of the coordinate frame have on the coordinates of a point P fixed in the plane.

Then we will determine the effects when the frame is fixed but the vector r connecting the origin O and the point P is rotated by an angle.



We first define the numbers using the right-triangle trig identities.

We adopt CCW as the positive direction.

Recall that cos alpha is the adjacent side of the triangle div by the hypotenuse

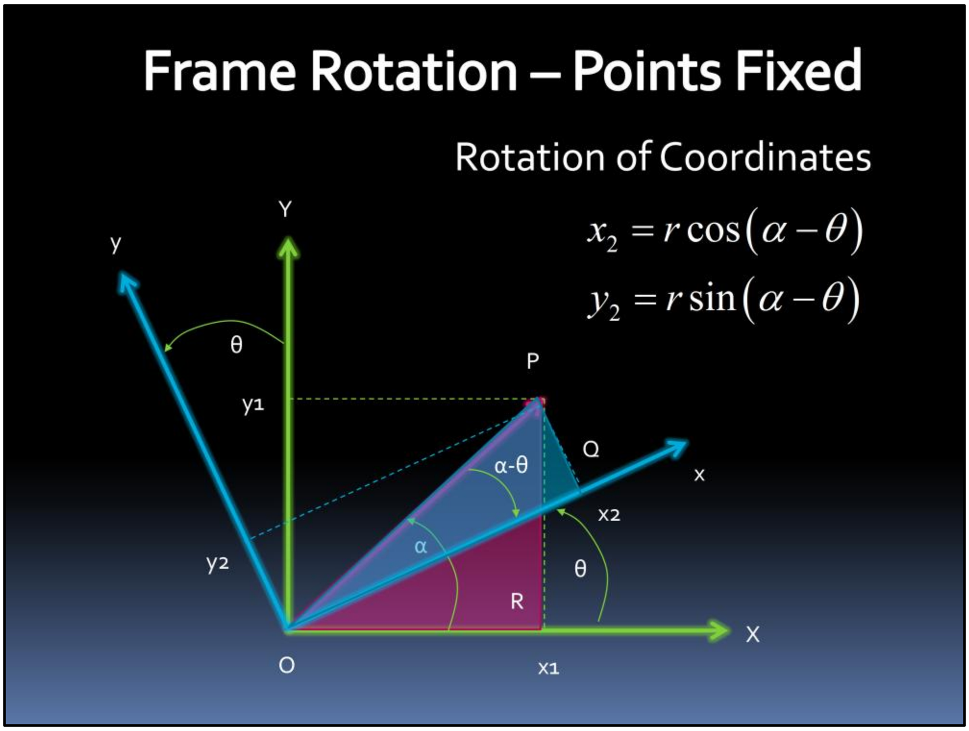
sin alpha is opposite side div by the hypotenuse

We identify P with the vector r directed from the origin O to P with coordinates (x1, y1) defined using the right triangle trig identities

Frame Rotation – Points Fixed

The point P or vector r are fixed with respect to the origin

There is a rotation of the frames about the origin through an angle θ



Triangle OPR satisfies
x1 = r cos α, y1 = r sin α

In the figure we note that the angle bet the vector r and the rotated x-axis is (α-θ).

In triangle OPQ then

x2 = r cos (α-θ). , y2 = r sin (α-θ). These expressions can be expanded:

x2 = r cos α cos θ + r sin α sin θ, y2 = r sin α cos θ – r cos α sin θ (next slide)

Giving the desired expression

x2 = x1 cos θ + y1 sin θ and y2 = y1 cos θ – x1 sin θ

Frame Rotation – Points Fixed

Rotation of Coordinates

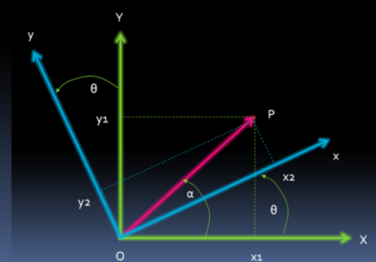
$$x_2 = \underline{r\cos(\alpha)}\cos(\theta) + \underline{r\sin(\alpha)}\sin(\theta)$$

$$y_2 = \underline{r\sin(\alpha)}\cos(\theta) - \underline{r\cos(\alpha)}\sin(\theta)$$

$$x_2 = x_1\cos(\theta) + y_1\sin(\theta)$$

$$y_2 = y_1\cos(\theta) - x_1\sin(\theta)$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$



Here we use the trig identities and the definition of x1 and y1 to reduce the expression and obtain the set of eq
for expressing the point according to a frame rotation (point fixed)

Those familiar with matrix multiplication can see that we can group the terms accordingly and the matrix plays the role of transforming the point in the old system

Point Rotation – Frame Fixed

Previously

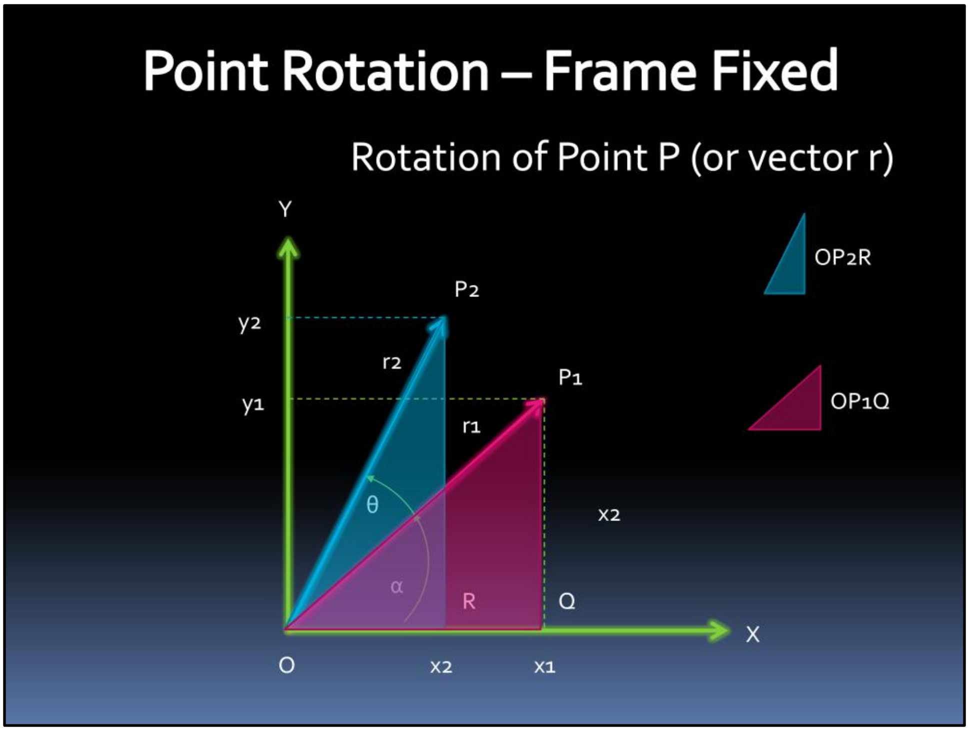
The point P or vector r was fixed with respect to the origin

There was a rotation of the frames about the origin through an angle θ

Next

The coordinate frame is fixed

The point P or vector r rotates about the origin through an angle θ from r_1 to r_2 – Notice that the length of a vector is invariant under rotation



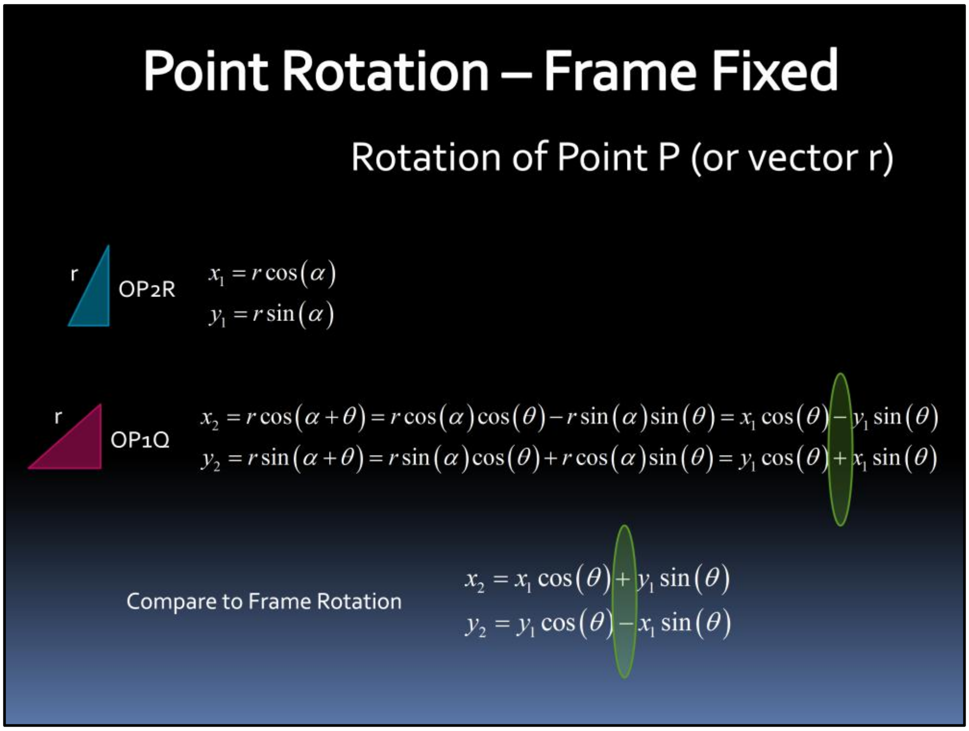
Previously we thought of the vector r or point P as being fixed as the frame rotated about the origin.
Now we can think of the frame as being fixed and instead rotate the vector about the origin through an angle theta from r1 to r2 (CCW positive)

The triangle OP1Q gives x1 = r cos alpha; y1 = r sin alpha

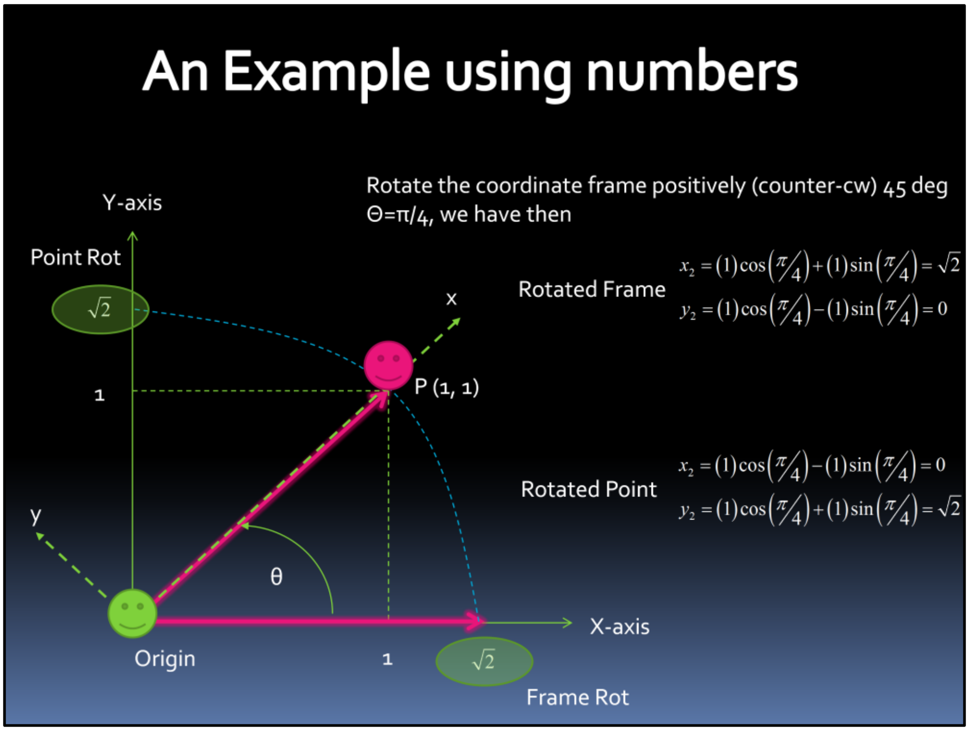
While in the triangle OP2R we have x1 = r cos (alpha + theta); y1 = r sin (alpha + theta)

Work out the trig identities (sum-differences formulae) and you get

x2 = x1 cos theta – y1 sin theta; y2 = y1 cos theta + x1 sin theta (next slide)



Notice here the different resulting signs in the two sets of equations



Let's give some values to the equations' params:
set theta = Pi/4 with P(1,1) and r = sqrt(2) in the original system

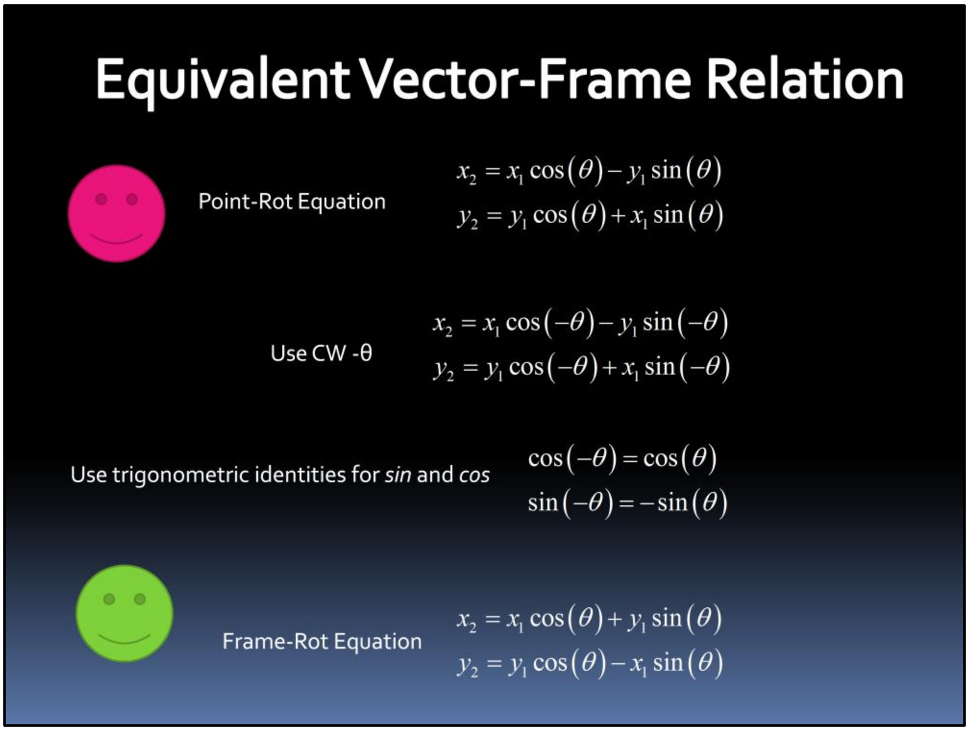
What are x2 and y2 according to the sets of eq above?

In the rotated coord frame x, y the point P lies on the x-axis with coord (sqrt(2), 0)

but if we however rotate the point P through an angle theta = Pi/4,
the resulting coord of P using the corresponding set of equations are (0, sqrt(2))

In the first case the perspective is that of an observer standing at the fixed point (pink face)

In the second case the observer is seated in the fixed frame (green face)



Recall sum-diff formulae

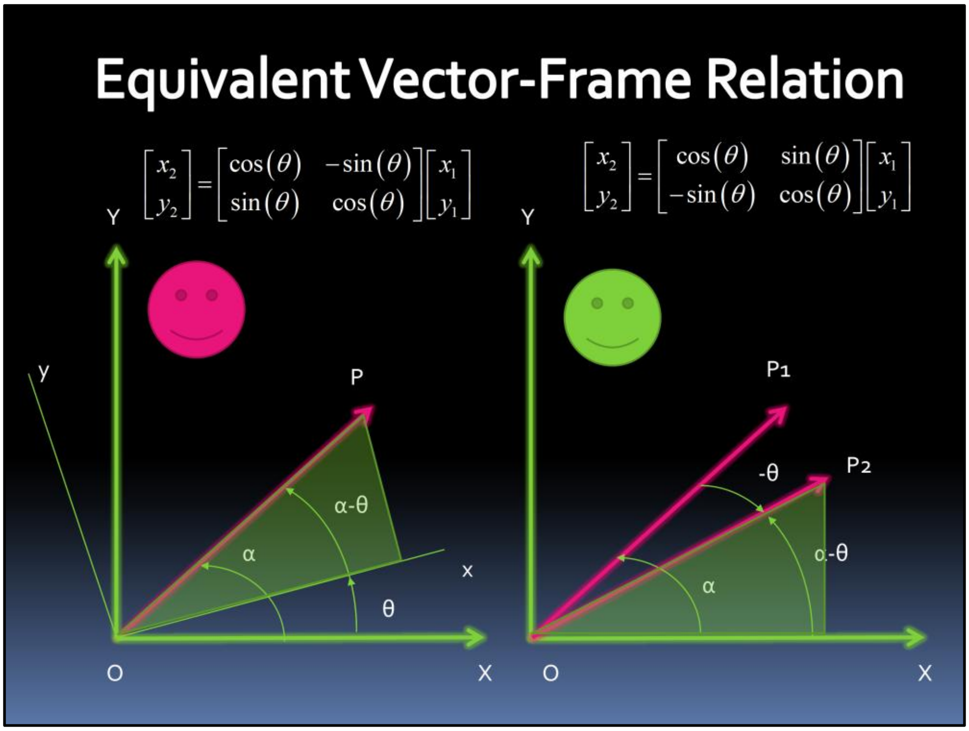
sin(u+/- v) = sin u cos v +/- cos u sin v

cos(u+/-v) = cos u cos v -/+sin u sin v

Recall trig identities

cos(-theta) = cos(theta)

sin(-theta)=-sin(theta)



Equivalent rotations:

We have just seen that two types of rotations (one in which the frame is fixed, and another in which the point is fixed) can produce very different results

However we will see that a CCW (positive) rotation of a coordinate frame through an angle +theta results in exactly the same vector-frame relationship as a (CW) rotation of the vector through an angle -theta

If the coordinate frame is rotated through a positive (CCW) angle theta = + Pi/4, the fixed vector (1,1) is transformed into the vector (sqrt(2),0) which lies along the new x-axis (relative to the fixed point guy)

If however, the vector (1,1) is rotated CW through the angle –Pi/4, relative to the FIXED coordinate frame XY, the resulting vector is again (sqrt(2),0) along the X-axis (relative to the fixed frame guy) frame

point

A

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

B

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$B=A^T$

$$B = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$\cos(-\theta) = \cos(\theta); \sin(-\theta) = -\sin(\theta)$$

$$B = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

The equivalence relation in matrix form

As a note: the sum and difference formulae for sine and cosine in matrix form:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha) & -\cos(\theta)\sin(\alpha) - \sin(\theta)\cos(\alpha) \\ \sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha) & -\sin(\theta)\sin(\alpha) + \cos(\theta)\cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{bmatrix}$$

These matrices form a representation of the rotational group in the plane (SO(2))
Pre-multiplying a vector by this matrix is the same as rotating it by the sum of the two angles

The composition of rotations from the sum-difference sine-cosine identity

Read Material on Matrix Operations

And run Matlab Scripts