Now onto to the 3D world

Rotations in 3 dimensions

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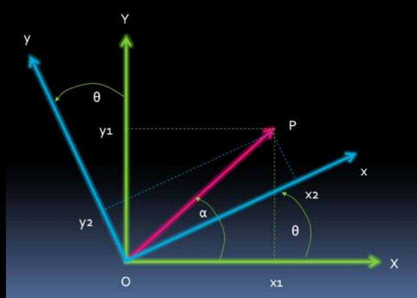
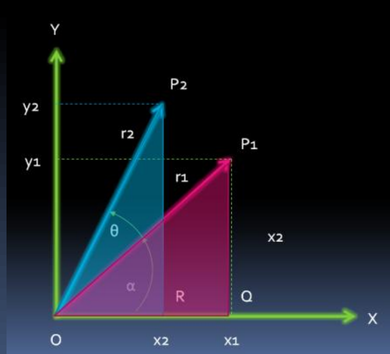
IMPS Fall 2011

RECAP

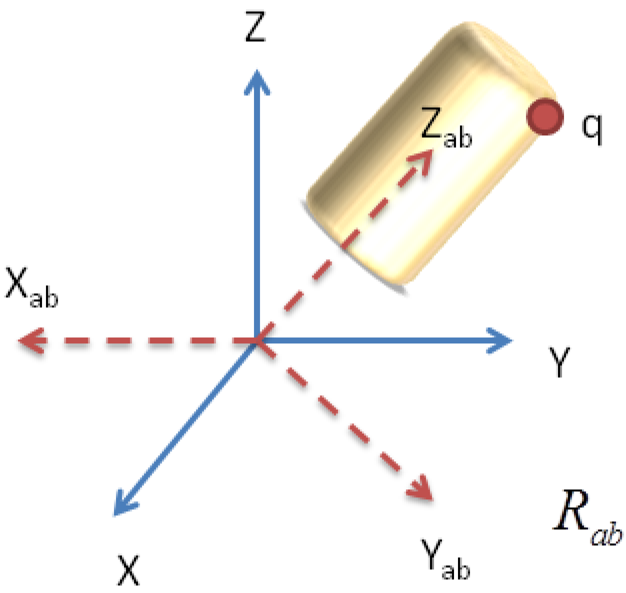
Rotate Frame



Rotate vector



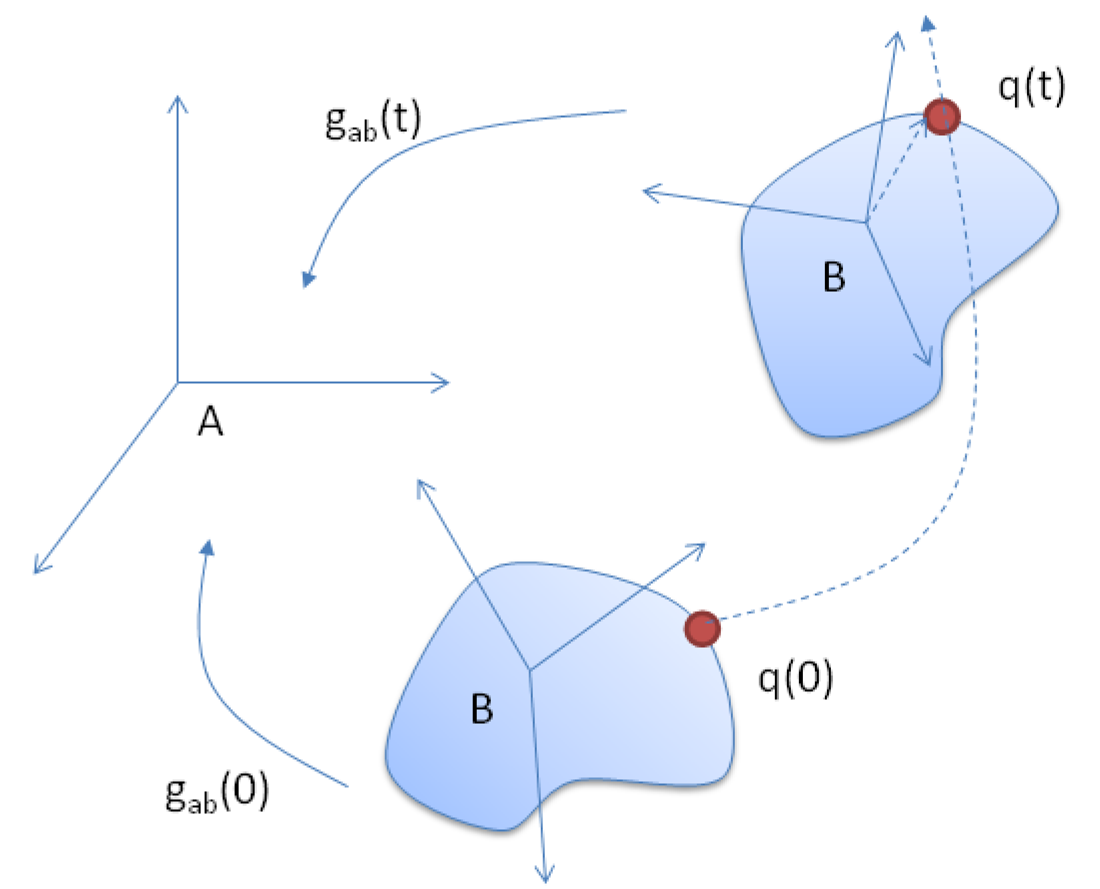
- Rotations are represented by matrices of the group SO(2)
- Determinant is 1, $A^{-1}=A^T$, $A.A^T=Id$
- Passive Convention: Rotations of a frame (fixed point)
- Active Convention: Rotations of a vector (fixed frame)
- One is the transpose of the other (for theta > 0)



$$R_{ab} = \left[X_{ab}, Y_{ab}, Z_{ab} \right] = \begin{pmatrix} x_{ab}^{X} & y_{ab}^{X} & z_{ab}^{X} \\ x_{ab}^{Y} & y_{ab}^{Y} & z_{ab}^{Y} \\ x_{ab}^{Z} & y_{ab}^{Z} & z_{ab}^{Z} \end{pmatrix}$$

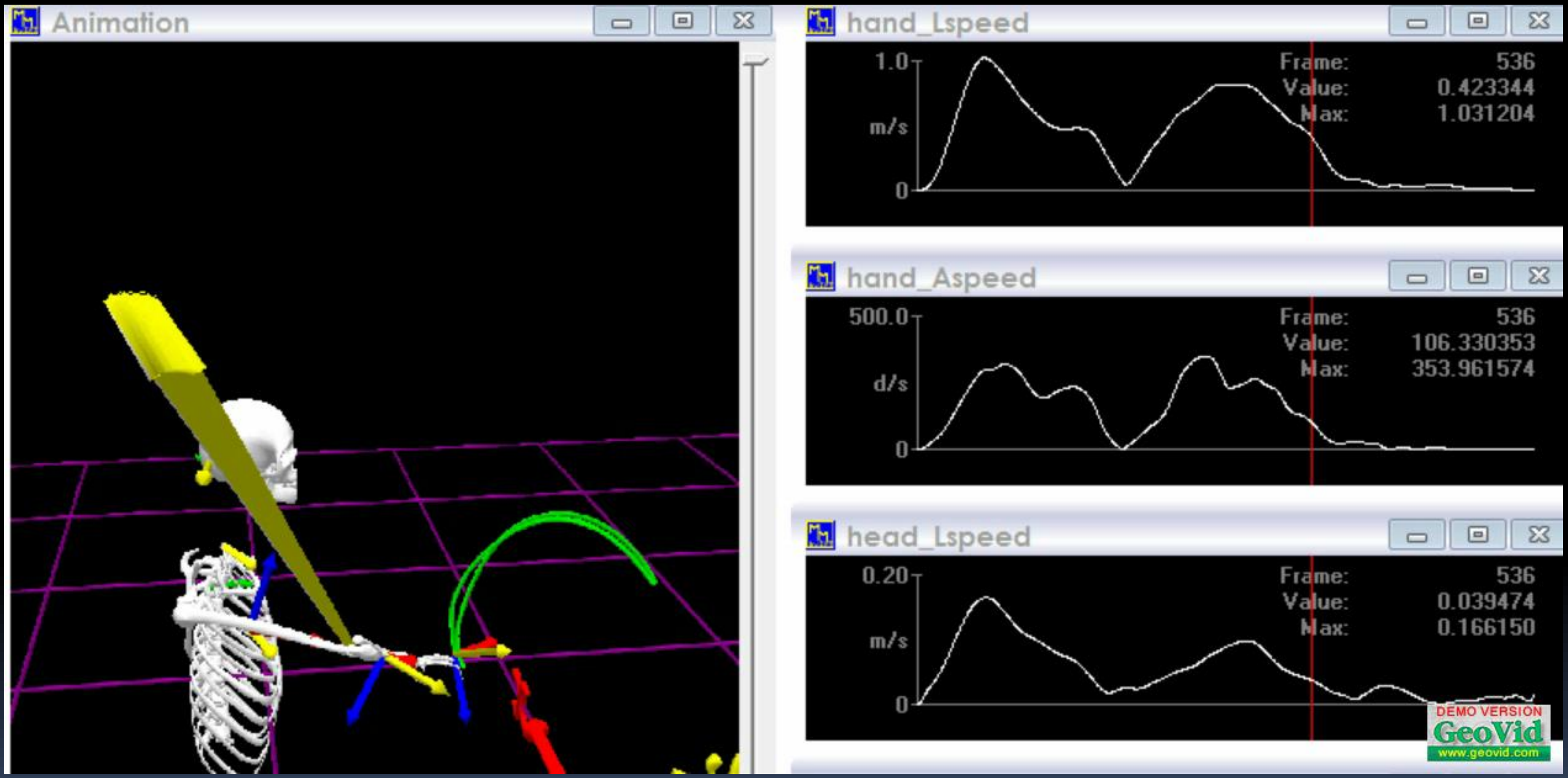
ROTATION OF A RIGID OBJECT ABOUT A POINT (DOTTED COORDINATE FRAME IS ATTACHED TO THE RIGID BODY

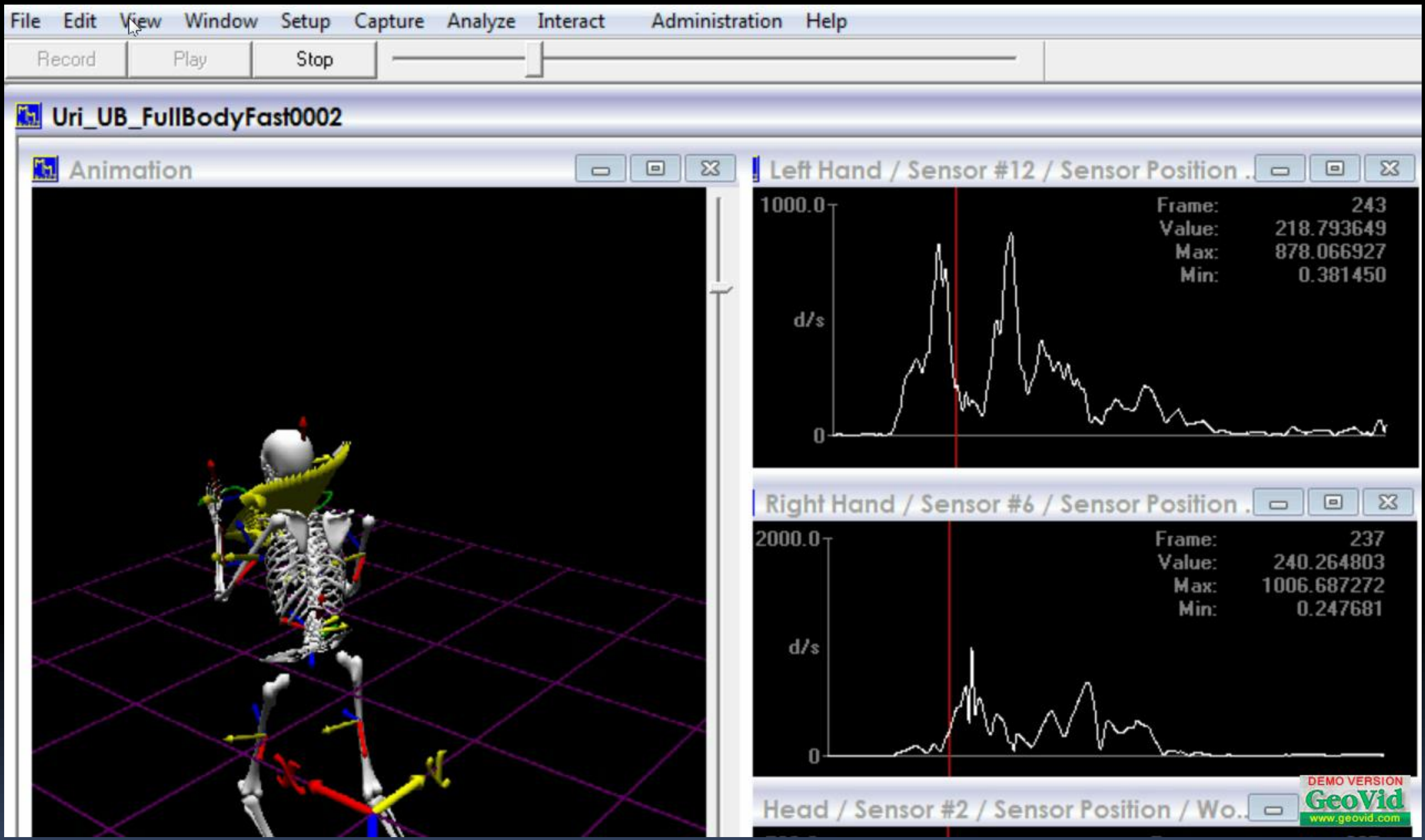
Can track the rotation path of the Rigid Body



TRAJECTORY OF A RIGID BODY RELATIVE TO A FIXED FRAME

Reaching around a physical obstacle





File Edit View Window Setup Capture Analyze Interact Administration Help
Record Play Stop
Uri_UB_FullBodyFast0002
Animation
Left Hand / Sensor #12 / Sensor Position ...
1000.0
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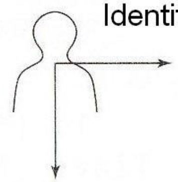
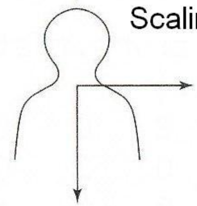
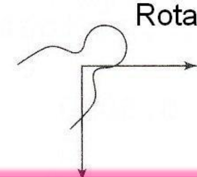
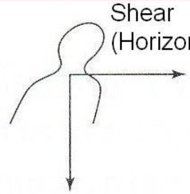
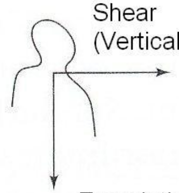
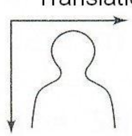
Frame: 243
Value: 218.793649
Max: 878.066927
Min: 0.381450
Right Hand / Sensor #6 / Sensor Position ...
2000.0
d/s
0
Frame: 237
Value: 240.264803
Max: 1006.687272
Min: 0.247681
Head / Sensor #2 / Sensor Position / Wo...
DEMO VERSION
GeoVid
www.geovid.com

Linear Transformations

Affine Matrix, T	Coordinate Equations	Diagram
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w$ $y = z$	Identity 
$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = s_x w$ $y = s_y z$	Scaling 
$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w\cos\theta - z\sin\theta$ $y = w\sin\theta + z\cos\theta$	Rotation 
$\begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w + \alpha z$ $y = z$	Shear (Horizontal) 
$\begin{bmatrix} 1 & \beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = w$ $y = \beta w + z$	Shear (Vertical) 
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix}$	$x = w + \delta_x$ $y = z + \delta_y$	Translation 

Special Orthogonal SO(2) Group

$A' = A^{-1}$

$\mathrm{Det}(A) = +1$

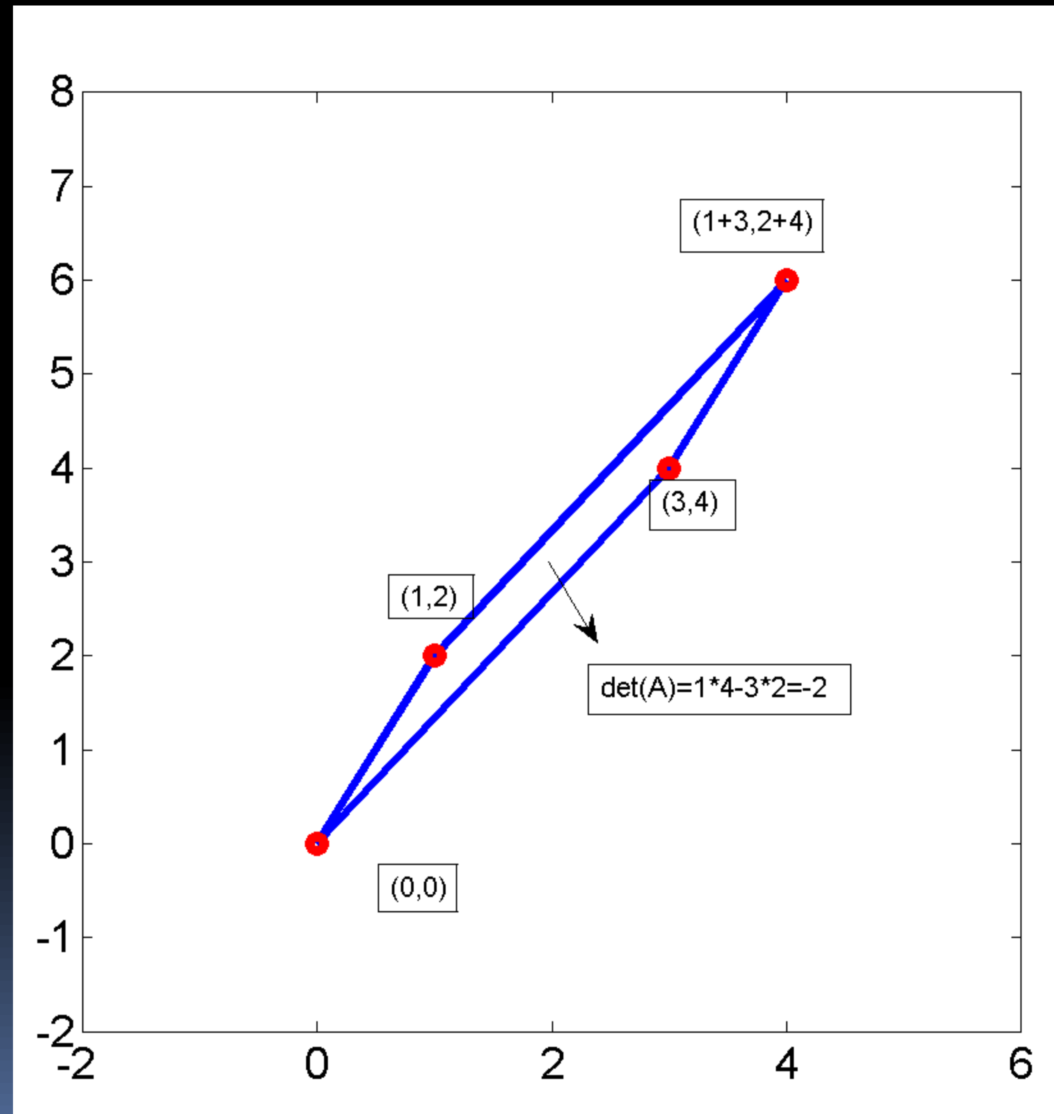
Determinant in 2D = Area inside the Parallelogram
From the row vectors of the matrix A

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1*4 - 3*2$$

$$R = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\det(R) = |R| = \cos(\theta)^2 + \sin(\theta)^2$$
$$= 1 = |R'|$$

Linear transformation of a square

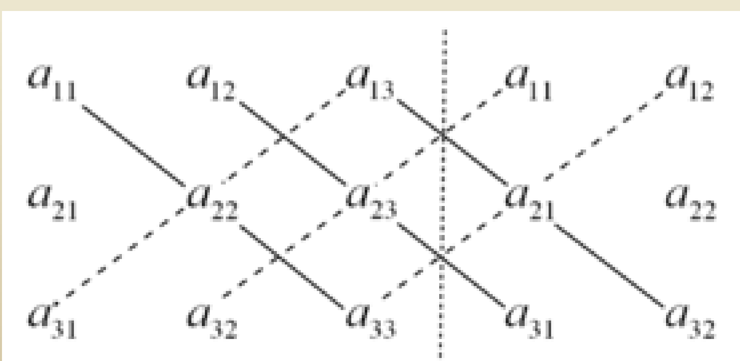


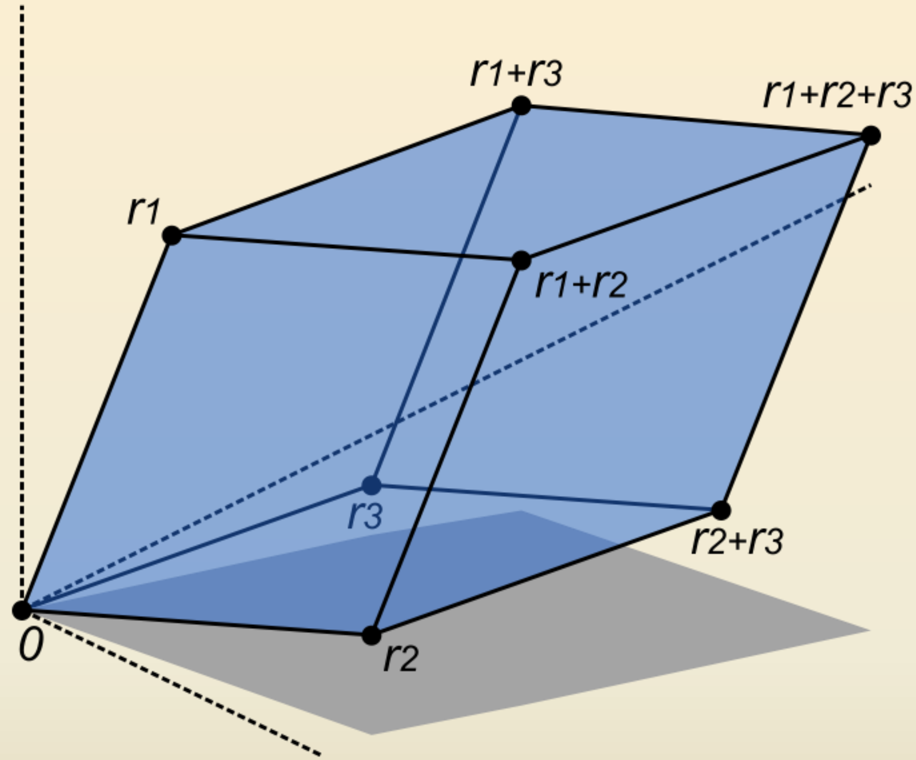
Determinant in 3D = Volume inside the Parallelepiped from the row vectors of the matrix A

$$\begin{matrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \\ \downarrow & \downarrow & \downarrow \end{matrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Rule of Sarrus



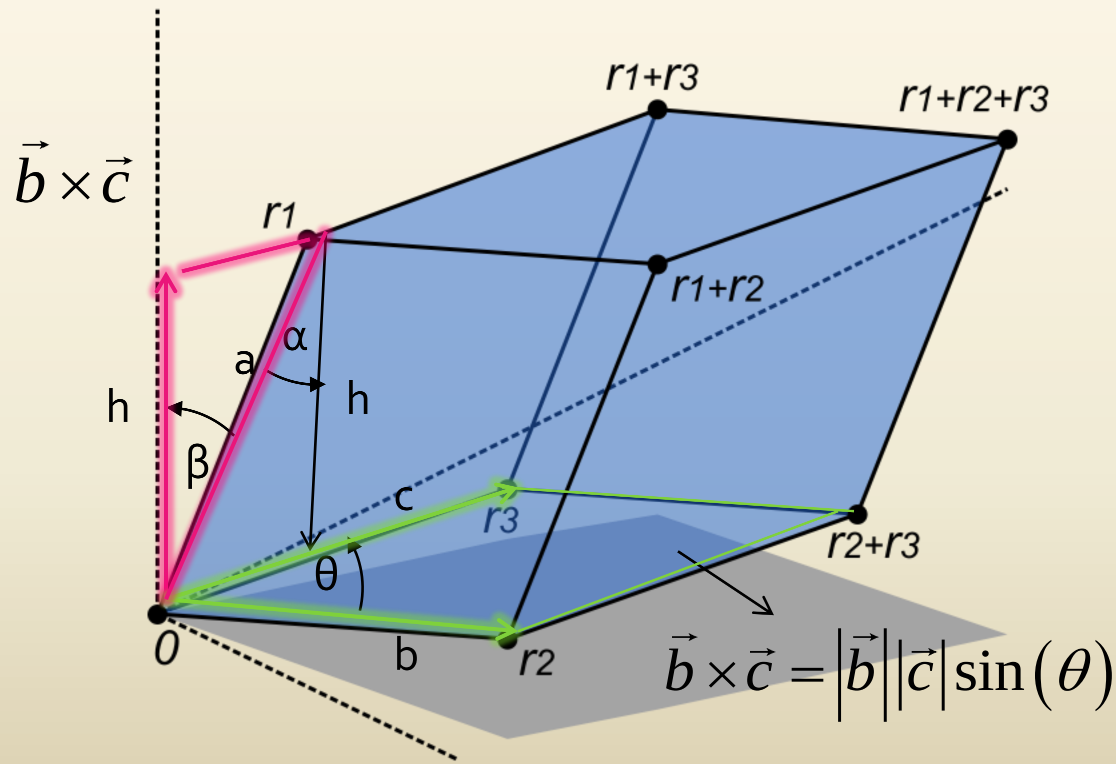


Geometric interpretation of the Determinant in the 3D case:
A linear transformation of a cube, then take the volume of that

$$V = Area \cdot h = |\vec{a}|\left|\vec{b} \times \vec{c}\right|\cos(\beta)$$

$$where \quad h = |\vec{a}|\cos(\beta)$$

$$V = \left\|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right\|$$



Recall that:

Dot product measures “parallelness”

Cross Product measures “perpendicularity”

Special Orthogonal SO(3) Group

- Determinant is +1 (Proper Rotations only)
- Transpose R = the Inverse R: $R \cdot R^T = Id$
- The product of two orthogonal matrices is orthogonal (all orthogonal matrices form a group O3)
- The set of proper rotations (det +1) is the SO(3) group
- The cols and rows of SO3 matrices are orthonormal

Three parameterizations of the SO(3) group

- Three Euler Angles

Leonard Euler
Swiss

- (Unit) Vector, Angle R(θ **n**)

- Euler-Olinde Rodrigues (Quaternions)

Sir Rowan Hamilton
Irish

Range of Angles

$$0 \le \omega < 2\pi$$

$$-\pi < \omega \le \pi$$

The identity lies in the middle of it

Conventions:

Active Convention (the vector moves, transform the vectors of the space)
Passive Convention (the frame moves, transform the frame)

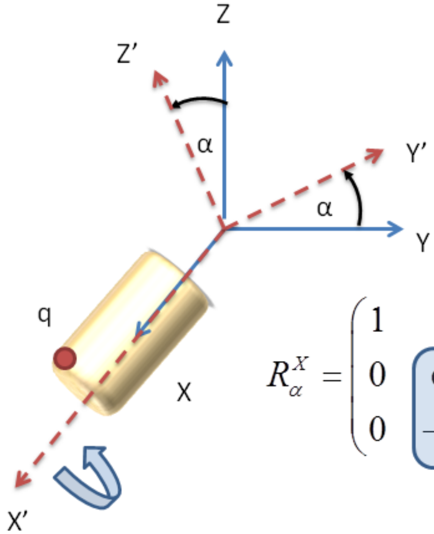
Notation:
α, β, γ

E.g. R(α, β, γ) = R(γ**z**) R(β**y**) R(α**x**)

$$0 \le \beta < 2\pi$$

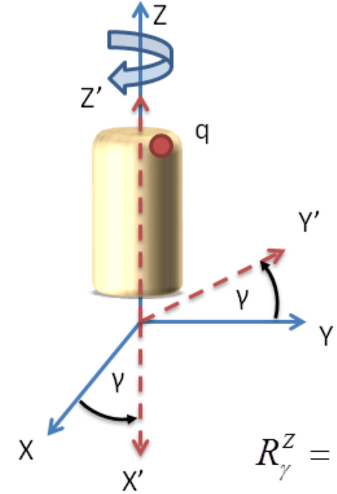
$$-\pi < \alpha, \gamma \le \pi$$

Euler Angles Orthogonal Matrices in SO(3)



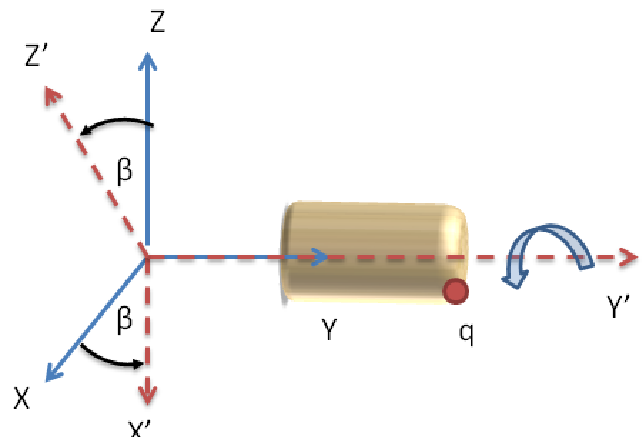
$$R_{\alpha}^{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Rotate around X by angle α



$$R_{\gamma}^{Z} = \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0 \\ -\sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotate around Z by angle γ



$$R_{\beta}^{Y} = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

Rotate around Y by angle β

Angle Vector Parameterization:

Statement of the Problem: Given a (rotation) matrix in SO(3) e.g. A(γ,z) find the angle of rotation γ about the unitary vector n

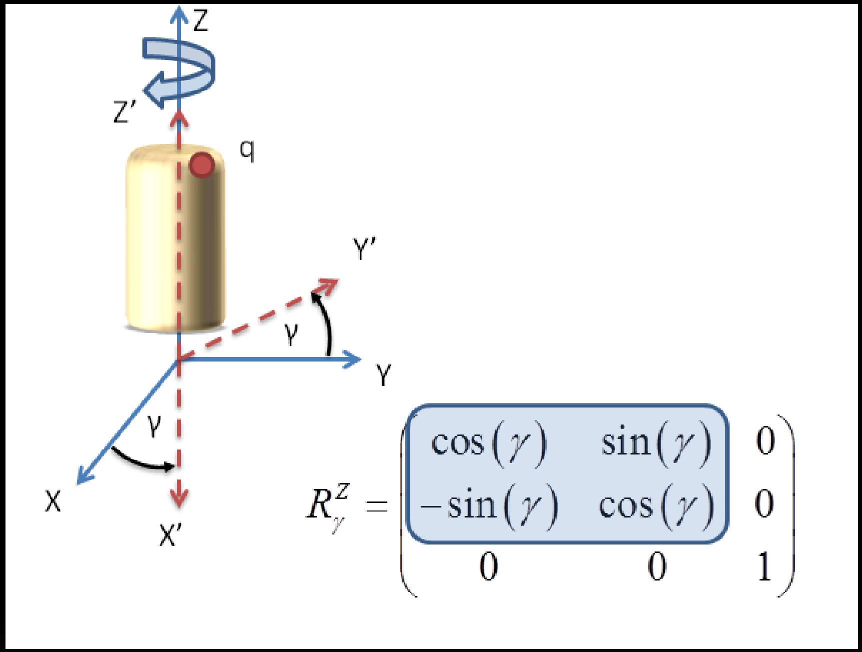
Unit Vector

$$\vec{n} = \left(n_x, n_y, n_z\right)$$

$$n_x = \frac{R_{32} - R_{23}}{2\sin\left(\gamma\right)}$$

$$n_y = \frac{R_{13} - R_{31}}{2\sin\left(\gamma\right)}$$

$$n_z = \frac{R_{21} - R_{12}}{2\sin\left(\gamma\right)}$$



Angle

$$\cos\left(\gamma\right) = \frac{1}{2} Tr\left(R\right) = \frac{1}{2}\left(R_{11} + R_{22} + R_{33}\right)$$